APR 22 '41 B

THE QUARTERLY JOURNAL OF MATHEMATICS.

OXFORD SERIES

Volume 11 *No.* 44 *December* 1940

CONTENTS

R. D. Misra: On a New Non-Differentiable Function . 225

R. Rado: The Distributive Law for Products of Infinite Series 229

E. Best: A Theorem on Hausdorff Measure . . . 243

J. L. Burchnall and **T. W. Chaundy**: Expansions of Appell's Double Hypergeometric Functions . 249

B. Spain: A Boundary Problem 271

G. N. Watson: An Inequality in the Triangle . . 273

M. L. Cartwright: On Level Curves of Integral Functions 277

A. E. Ingham: On the Estimation of $N(\sigma, T)$ . . 291

A. Erdélyi: On Fractional Integration and its Application to the Theory of Hankel Transforms . . 293

OXFORD
AT THE CLARENDON PRESS
1940

Price 7s. 6d. net

PRINTED IN GREAT BRITAIN BY JOHN JOHNSON AT THE OXFORD UNIVERSITY PRESS

THE QUARTERLY JOURNAL OF MATHEMATICS

OXFORD SERIES

Edited by T. W. CHAUNDY, U. S. HASLAM-JONES, J. H. C. THOMPSON

With the co-operation of A. L. DIXON, W. L. FERRAR, G. H. HARDY, E. A. MILNE, E. C. TITCHMARSH

THE QUARTERLY JOURNAL OF MATHEMATICS (OXFORD SERIES) is published in March, June, September, and December, at a price of 7s. 6d. net for a single number with an annual subscription (for four numbers) of 27s. 6d. post free.

Papers, of a length normally not exceeding 20 printed pages of the Journal, are invited on subjects of Pure and Applied Mathematics, and should be addressed 'The Editors, Quarterly Journal of Mathematics, Clarendon Press, Oxford'. Contributions can be accepted in French and German, if in typescript (formulae excepted). Authors of papers printed in the Quarterly Journal will be entitled to 50 free offprints. Correspondence on the *subject-matter* of the Quarterly Journal should be addressed, as above, to 'The Editors', at the Clarendon Press. All other correspondence should be addressed to the Publisher (Humphrey Milford, Oxford University Press, at the temporary address).

HUMPHREY MILFORD
OXFORD UNIVERSITY PRESS
AMEN HOUSE, LONDON, E.C.4

Temporary address:
SOUTHFIELD HOUSE, HILL TOP ROAD, OXFORD

ON A NEW NON-DIFFERENTIABLE FUNCTION

By R. D. MISRA (*Lucknow*)

[Received 13 March 1940; in revised form 15 May 1940]

In my paper* 'On Hilbert's Curve' I have obtained the analytical representation of the space-filling curve given by Hilbert.† It was indicated in that paper that the functions $f_1(t)$ and $f_2(t)$ which occur in the parametric representation of Hilbert's curve are non-differentiable functions of t. They are quite different from any non-differentiable function defined by previous writers. The purpose of this paper is to supply the proof of the non-differentiability of this new function. I should mention that the notation used in the definitions of the functions is partly different from that used in my previous paper.

1. Analytical representation

If $t = \cdot a_1 a_2 \dots a_m \dots$, where the a's are either zero or 1 and the base of the radix fraction is 2, then t represents for different values of a's all the points rational and irrational of the closed interval $(0, 1)$. Hilbert's space-filling curve (x, y) which passes through all the points of a unit square is represented by

$$x = f_1(t) = \cdot b_1 b_2 \dots b_m \dots \tag{1.1}$$

$$y = f_2(t) = \cdot c_1 c_2 \dots c_m \dots \tag{1.2}$$

where $0 \leqslant t \leqslant 1$ and the b's and c's are either $+1$ or -1 according to the a's. The relationship between the a's, b's, and c's is given below, the base of the radix fraction in the representation of the latter also being 2.

The following notation will be used throughout:

(α) $[r]$ is the largest integer not exceeding r,

(β) $\sigma_n = 1+[\frac{1}{2}a_1+\frac{1}{2}a_2]+\dots+[\frac{1}{2}a_{2n-5}+\frac{1}{2}a_{2n-4}]$,

(γ) $A_{2n} = \begin{cases} 0 & (a_1+a_2+\dots+a_{2n-2} \text{ even}), \\ a_{2n} & (a_1+a_2+\dots+a_{2n-2} \text{ odd}). \end{cases}$

* *Proc. Benares Math. Soc.* 16 (1934), 13–34.

† *Math. Annalen*, 38 (1891), 459.

We now define

$$\left.\begin{aligned} b_1 &= 1 \\ b_{2m} &= (-1)^{\sigma_{2m}+a_{4m-2}+A_{4m-2}} \\ b_{2m+1} &= (-1)^{\sigma_{2m+1}+a_{4m-1}+a_{4m}-A_{4m}} \end{aligned}\right\} \tag{1.3}$$

and

$$\left.\begin{aligned} c_1 &= 1 \\ c_{2m} &= (-1)^{\sigma_{2m}+a_{4m-2}+a_{4m-2}-A_{4m-2}} \\ c_{2m+1} &= (-1)^{\sigma_{2m+1}+a_{4m-1}+A_{4m}} \end{aligned}\right\}. \tag{1.4}$$

I have shown in my former paper that $f_1(t)$ and $f_2(t)$ are uniquely determined, though for rational values of t we have two different representations of t by dual fractions.

2. Continuity of the function $f(t)$

It is easy to see that the function $f(t)$ is a continuous function of t. For, if we take two points t_1 and t_2 which are very near to each other differing in the $2m$th term of the radix fraction, then the corresponding $f(t_1)$ and $f(t_2)$ do not differ up to the $(m-1)$th term. Hence the difference between $f(t_1)$ and $f(t_2)$ can be made as small as we please by making the difference between t_1 and t_2 suitably small, which proves the proposition.

3. Non-differentiability of the function $f(t)$

(a) *Non-differentiability at the set of points t with double representation.*

Of the two representations of t one will be ending and the other non-ending. Let the ending one be

$$t = \cdot a_1 a_2 \ldots a_r\, 0\, 0 \ldots . \tag{3.1}$$

Let us for simplicity assume for the moment that $\sum_{p=1}^{r} a_p$ is even and let m be any odd integer such that $2m > r$.

We have then

$$a_{2m} = 0, \qquad a_{2m+p} = 0,$$

and

$$\sigma_{m+1} = \sigma_{m+1+p}, \qquad A_{2m} = A_{2(m+p)} = 0.$$

We further notice from (1.3) that, whereas every a_{4p} occurs explicitly in one of the equations giving the b's, the a_{4p-2} do not occur explicitly at all in any one of them; and, as all $a_{2(m+p)}$ are zero, a change in any one $a_{2(m+p)}$ will affect neither the σ's nor the A's. Therefore it is obvious that an addition to t of $2^{-(2m+4s)}$ (s being any non-negative integer), i.e. changing a_{2m+4s} from 0 to 1, will affect none

of the b's. Thus, if we take a sequence of values for $h_s = 2^{-2m-4s}$, s ranging from 0 to ∞, then $x'-x$ will always be zero, and therefore

$$\lim_{h_s \to 0} \frac{x'-x}{h_s} = 0. \tag{3.2}$$

On the other hand, if $2^{-2m-2-4s}$ be added to t, the σ's and A's will remain unchanged; but, since

$$b_{m+2+2s} = (-1)^{\sigma_{m+2+2s}+a_{2m+1}+4s+a_{2m+2}+4s-A_{2m+2}+4s}, \tag{3.3}$$

b'_{m+2+2s} will have the sign opposite to that of b_{m+2+2s} with other b's remaining unaltered. Therefore, if we take the sequence

$$h'_s = 2^{-2m-2-4s},$$

we get

$$\lim_{h'_s \to 0} \frac{x''-x}{h'_s} = \lim_{s \to \infty} \frac{\pm 2/2^{m+2+2s}}{1/2^{2m+2+4s}} \tag{3.4}$$

$$= +\infty \text{ or } -\infty \text{ according to the } a\text{'s}. \tag{3.5}$$

If $\sum_{p=1}^{r} a_p$ be odd, then the first case gives the limit $+\infty$ or $-\infty$, while the second gives zero. If m be even, then also, as one can easily see, the two limits will be different, being 0 and $+\infty$ or $-\infty$.

These results prove that the function $f_1(t)$ has no progressive derivate and therefore no differential coefficient at any rational value of t.

(b) *Non-differentiability at the set of points t with unique representation.*

Since, in this case, at no stage can the subsequent a's be all 0 or all 1, the representation of t may belong to one of the following classes:

(i) 0 and 1 are not arranged in any regular order;

(ii) the zero a's are inserted between the unit a's in some order;

(iii) the unit a's are inserted between the zero a's in some order.

Class (i). From the analysis of case (a) we easily see that, if any a_{2m} be zero with m and $\sum_{p=1}^{2(m-1)} a_p$ odd, then the addition of 2^{-2m} to t changes a_{2m} from 0 to 1 and b_{m+1} from $+1$ to -1 (or from -1 to $+1$). All the b's before b_{m+1} remain unaltered. Subsequent b's may change, but, as they would not all be $+1$ or -1, the change in the sign of b_{m+1} will certainly change x. Thus $(x'-x)/(t'-t)$ will be large and positive or negative, according to the value of b_{m+1}. Now as the zero a's and the unit a's are not arranged in any regular

order, we may choose two different sequences 2^{-2m-4s} and 2^{-2m-4q} such that

$$\lim_{h_s \to 0} \frac{x'-x}{t'-t} = -\infty \text{ or } +\infty$$

and

$$\lim_{h_q \to 0} \frac{x''-x}{t''-t} = +\infty \text{ or } -\infty,$$

showing that no progressive derivate exists for values of t belonging to class (i).

Class (ii). There are two possibilities: the zeros may be arranged in such a way that it is impossible to get the two required sequences h_s and h_q; or none of the a_{2m}'s may be zero.

To save space I refrain from giving a detailed discussion of these cases and merely mention that, in both these alternatives, one could easily get different left-hand or right-hand limits by subtracting or adding suitable sequences of the type $2^{-2m\pm1}$.

Class (iii). In this case one may have to add or subtract such different sequences as 2^{-2m-1} and $2^{-2m+1}+2^{-2m-1}$ to show that either the progressive or the regressive derivate does not exist. The case of $f_2(t)$ is similar to $f_1(t)$.

The significance of the non-existence of the progressive or the regressive derivate is to show that the function $f(t)$ is not only nowhere differentiable but also has no cusps.

I take this opportunity of expressing my sincere thanks to Dr. A. Erdélyi for his helpful advice ungrudgingly given on more than one occasion.

THE DISTRIBUTIVE LAW FOR PRODUCTS OF INFINITE SERIES

By R. RADO (*Sheffield*)

[Received 13 August 1940]

Introduction. A natural extension of the distributive law of multiplication

$$(X_1+X_2)(Y_1+Y_2) = X_1Y_1+X_1Y_2+X_2Y_1+X_2Y_2$$

would be an equation of the form

$$\Big(\sum_{\kappa=1}^{\infty} X_\kappa\Big)\Big(\sum_{\lambda=1}^{\infty} Y_\lambda\Big) = X_{\kappa_1}Y_{\lambda_1}+X_{\kappa_2}Y_{\lambda_2}+\dots. \tag{1}$$

Cauchy's and Dirichlet's methods of multiplying infinite series defined respectively by

$$\Big(\sum_{\kappa=1}^{\infty} X_\kappa\Big)\Big(\sum_{\lambda=1}^{\infty} Y_\lambda\Big) = \begin{cases} \displaystyle\sum_{\nu=2}^{\infty}\Big(\sum_{\lambda=1}^{\nu-1} X_{\nu-\lambda}Y_\lambda\Big) & (2) \\ \displaystyle\sum_{\nu=1}^{\infty}\Big(\sum_{\lambda\mid\nu} X_{\nu/\lambda}Y_\lambda\Big) & (3) \end{cases}$$

may be considered as combinations of the distributive and associative law, and it is well known that neither (2) nor (3) is true for every pair of convergent series $\sum X_\kappa$, $\sum Y_\lambda$.

We say a sequence $(\kappa_\nu, \lambda_\nu)$ $(\nu = 1, 2,\dots)$ of pairs of positive integers is *distributive* if the convergence of the two series on the left-hand side of (1) implies the convergence of the series on the right and the validity of the equation (1). In this note we determine all distributive sequences $(\kappa_\nu, \lambda_\nu)$. The problem of finding distributive sequences was stated by K. Knopp.†

Putting $X_{\kappa'} = Y_{\lambda'} = 1$ and $X_\kappa = Y_\lambda = 0$ $(\kappa \neq \kappa'; \lambda \neq \lambda')$ we obtain a necessary condition for the validity of (1), namely, that for exactly one suffix ν the equations $\kappa_\nu = \kappa'$; $\lambda_\nu = \lambda'$ hold. We may therefore restrict ourselves to sequences $(\kappa_\nu, \lambda_\nu)$ which contain every pair (κ', λ') of positive integers exactly once. Let us call such sequences (two-dimensional) *permutation sequences*.

It is easily seen that the following permutation sequence $(\kappa_\nu, \lambda_\nu)$ is distributive:

$$(1,1), (2,1), (2,2), (1,2), (3,1), (3,2), (3,3), (2,3), (1,3), (4,1),\dots. \tag{4}$$

† K. Knopp, *Theorie und Anwendungen der unendlichen Reihen* (1924), § 17.

On the other hand, if in Cauchy's and Dirichlet's multiplication formulae (2), (3) we write out explicitly the sums $\sum_{\nu}$ and omit the brackets, we are led to the permutation sequences

$$(1,1),\ (2,1),\ (1,2),\ (3,1),\ (2,2),\ (1,3),\ldots \tag{5}$$

$$(1,1),\ (2,1),\ (1,2),\ (3,1),\ (1,3),\ (4,1),\ (2,2),\ (1,4),\ldots \tag{6}$$

characterized respectively by

$$\kappa_\nu+\lambda_\nu \leqslant \kappa_{\nu+1}+\lambda_{\nu+1},$$

$$\kappa_\nu\lambda_\nu \leqslant \kappa_{\nu+1}\lambda_{\nu+1}.$$

The sequences (5) and (6) are not distributive.

For a given permutation sequence $(\kappa_\nu, \lambda_\nu)$ let P_ν be the point in a plane whose coordinates are κ_ν, λ_ν. By *rectangle* we understand a set of points (κ, λ) with positive integral coordinates κ, λ which is defined by inequalities

$$\alpha < \kappa \leqslant \beta, \qquad \gamma < \lambda \leqslant \delta.$$

Here α, β, γ, δ are non-negative integers, and $\alpha < \beta$; $\gamma < \delta$. For any fixed n let R_n be the minimum number of rectangles which can be found such that their sum set is $\{P_1, P_2,\ldots, P_n\}$.† We shall prove below that *a permutation sequence* $(\kappa_\nu, \lambda_\nu)$ *is distributive if and only if* R_n *is bounded as* $n \to \infty$ (Theorem A). One readily verifies that in the case of the sequence (4) we have $R_n \leqslant 2$ for every n while, in the case of (5) and (6), $R_n \to \infty$. Theorem A may be considered as a generalization of a theorem of E. Borel‡ which reads as follows. *A rearrangement* $\kappa_1, \kappa_2,\ldots$ *of the sequence of all positive integers has the property that the convergence of* $\sum_{\kappa} X_\kappa$ *implies the convergence of* $\sum_{\nu} X_{\kappa_\nu}$ *and the validity of*

$$\sum_{\kappa} X_\kappa = \sum_{\nu} X_{\kappa_\nu}$$

if and only if the number of blocks of consecutive numbers forming the set $\{\kappa_1, \kappa_2,\ldots, \kappa_n\}$ *is* $O(1)$ *as* $n \to \infty$. Theorem B below asserts that *distributive sequences* $(\kappa_\nu, \lambda_\nu)$ *are also characterized by the property that the partial sums of the series on the right of* (1) *are bounded whenever the same is true for the two series on the left.*

† Throughout we denote by $\{A, B,\ldots, K\}$ the set whose elements are $A, B,\ldots, K$, and sums, differences, products of sets have their usual meanings; 0 stands for the empty set as well as for the number zero.

‡ E. Borel, *Bull. des sci. math.* (2), 14 (1890), 97.

It is easy to extend the proof of the sufficiency of the condition $R_n = O(1)$ to the case of products of any number r of series. The rectangles have to be replaced by r-dimensional 'rectangles'. However, for $r > 2$, the proof of the necessity of a condition of this kind seems to present difficulties. In the present proof (for $r = 2$) we make use of the following combinatorial theorem of F. P. Ramsay:†

Given three positive integers k, l, m, there is a positive integer $f = f(k, l, m)$ which has the following property. Whenever all $\binom{f}{l}$ systems of l numbers out of $1, 2, \ldots, f$ are arbitrarily divided into k classes there are always m distinct numbers $a_1, a_2, \ldots, a_m \leqslant f$ such that all systems of l numbers out of the a_μ's belong to the same class.

1. With the exception of $\sum, \prod$ all letters denote positive integers unless the contrary is explicitly stated. 'Number' means 'positive integer'. Instead of considering the set $\{P_1, P_2, \ldots, P_n\}$, where P_ν is the point with coordinates κ_ν, λ_ν, we argue more conveniently in terms of what is in effect the characteristic function of this set. The arguments are best followed with the help of diagrams.

Corresponding to every permutation sequence $(\kappa_\nu, \lambda_\nu)$ we define numbers $(\kappa, \lambda)^{(n)} = \pm 1$ as follows: $(\kappa, \lambda)^{(n)} = 1$ if (κ, λ) occurs among the pairs $(\kappa_1, \lambda_1), (\kappa_2, \lambda_2), \ldots, (\kappa_n, \lambda_n)$; and $(\kappa, \lambda)^{(n)} = -1$ for all remaining triples κ, λ, n.

Put

$$[\kappa, \lambda]^{(n)} = (\kappa, \lambda)^{(n)}(\kappa+1, \lambda)^{(n)},$$

$$[\kappa, \lambda]'^{(n)} = (\kappa, \lambda)^{(n)}(\kappa, \lambda+1)^{(n)},$$

$$S^{(n)} = \sum_{\kappa} \left(1 - \prod_{\lambda} \tfrac{1}{2}(1+[\kappa, \lambda]^{(n)})\right),$$

$$S'^{(n)} = \sum_{\lambda} \left(1 - \prod_{\kappa} \tfrac{1}{2}(1+[\kappa, \lambda]'^{(n)})\right),$$

$$m^{(n)} = \max_{\nu \leqslant n}(\kappa_\nu + \lambda_\nu).$$

Then

$$\lim_{n \to \infty} m^{(n)} = \infty.$$

If $\kappa \geqslant m^{(n)}$, then, for any λ,

$$\kappa + \lambda > m^{(n)}; \qquad (\kappa, \lambda)^{(n)} = -1; \qquad [\kappa, \lambda]^{(n)} = 1.$$

More generally, we have

$$[\kappa, \lambda]^{(n)} = [\kappa, \lambda]'^{(n)} = 1 \tag{7}$$

† F. P. Ramsay, *Proc. London Math. Soc.* (2), 30 (1930), 267, Theorem B. Alternative proofs were given by Th. Skolem, *Fund. Math.* 20 (1933), 257, Satz 3, and by P. Erdős and G. Szekeres, *Compositio Math.* 2 (1935), 464.

provided that $\max(\kappa, \lambda) \geqslant m^{(n)}$.

Therefore $0 < S^{(n)},\ S'^{(n)} < m^{(n)}$.

Consider, for some fixed n, λ_0, all κ for which $[\kappa, \lambda_0]^{(n)} = -1$. Suppose that these are $\kappa = \alpha_1, \alpha_2, \ldots, \alpha_r$ where $\alpha_1 < \alpha_2 < \ldots < \alpha_r$. Then, if $r \geqslant 2$,

$$(\alpha_\rho, \lambda_0)^{(n)} = -(\alpha_{\rho+1}, \lambda_0)^{(n)} \quad (\rho < r). \tag{8}$$

THEOREM A. *A permutation sequence $(\kappa_\nu, \lambda_\nu)$ is distributive if and only if*

$$S^{(n)} + S'^{(n)} = O(1) \quad \text{as } n \to \infty. \tag{9}$$

Let us first show that (9) is equivalent to saying that $\{P_1, P_2, \ldots, P_n\}$ is the sum set of $O(1)$ rectangles. It is immaterial whether the rectangles are or are not allowed to overlap. Suppose, first of all, that r rectangles can be found whose sum is $\{P_1, P_2, \ldots, P_n\}$, i.e. that, for some n, r, α_ρ, β_ρ, γ_ρ, δ_ρ satisfying $\alpha_\rho < \beta_\rho$; $\gamma_\rho < \delta_\rho$ (some of the α_ρ, γ_ρ may be zero),

$$\sum_{\nu \leqslant n} \{(\kappa_\nu, \lambda_\nu)\} = \sum_{\rho \leqslant r} \sum_{\substack{\alpha_\rho < \kappa \leqslant \beta_\rho \\ \gamma_\rho < \lambda \leqslant \delta_\rho}} \{(\kappa, \lambda)\}.$$

Then, obviously, if κ' differs from every β_ρ, we have, for any λ,

$$(\kappa', \lambda)^{(n)} = (\kappa'+1, \lambda)^{(n)}; \qquad [\kappa', \lambda]^{(n)} = 1.$$

Similarly, when λ' differs from every δ_ρ,

$$[\kappa, \lambda']'^{(n)} = 1$$

for all κ. For such κ', λ', we have

$$\prod_\lambda \tfrac{1}{2}(1+[\kappa', \lambda]^{(n)}) = \prod_\kappa \tfrac{1}{2}(1+[\kappa, \lambda']'^{(n)}) = 1,$$

and therefore $S^{(n)} + S'^{(n)} \leqslant 2r$.

Next, let us assume that, for some n,

$$S^{(n)} + S'^{(n)} \leqslant c.$$

Let $\kappa_2', \kappa_3', \ldots, \kappa_p'$ be all numbers κ for which

$$\prod_\lambda \tfrac{1}{2}(1+[\kappa, \lambda]^{(n)}) = 0,$$

and $\lambda_2', \lambda_3', \ldots, \lambda_q'$ be all numbers λ for which

$$\prod_\kappa \tfrac{1}{2}(1+[\kappa, \lambda]'^{(n)}) = 0,$$

and suppose that

$$\kappa_2' < \kappa_3' < \ldots < \kappa_p'; \qquad \lambda_2' < \lambda_3' < \ldots < \lambda_q'.$$

Then $p, q \geqslant 2$, $\quad (p-1)+(q-1) = S^{(n)} + S'^{(n)} \leqslant c.$

If κ differs from all numbers κ'_μ and λ is arbitrary, then $[\kappa,\lambda]^{(n)}=1$, i.e. $(\kappa,\lambda)^{(n)}=(\kappa+1,\lambda)^{(n)}$. A similar statement holds for the second coordinate. Hence, putting $\kappa'_1=\lambda'_1=0$,

$$\sum_{\nu\leqslant n}\{(\kappa_\nu,\lambda_\nu)\}+\{(0,0)\}$$
$$=\sum_{\substack{\mu<p\\ \nu<q}}\sum_{\substack{\kappa'_\mu<\kappa\leqslant\kappa'_{\mu+1}\\ \lambda'_\nu<\lambda\leqslant\lambda'_{\nu+1}}}\{(\tfrac{1}{2}(1+(\kappa'_{\mu+1},\lambda'_{\nu+1})^{(n)})\kappa,\tfrac{1}{2}(1+(\kappa'_{\mu+1},\lambda'_{\nu+1})^{(n)})\lambda)\}+\{(0,0)\}. \tag{10}$$

This result shows that $\{P_1, P_2,..., P_n\}$ is the sum of not more than

$$(p-1)(q-1)\ (\leqslant c^2)$$

non-overlapping rectangles.

2. In this section we prove the sufficiency of the condition (9). Suppose that a permutation sequence satisfies, for every n,

$$S^{(n)}+S'^{(n)}\leqslant C,$$

where C is independent of n. Let $\sum_\kappa X_\kappa$, $\sum_\lambda Y_\lambda$ be two convergent series (X_κ, Y_λ complex). We have to prove (1). Put

$$s_n=X_1+...+X_n;\qquad t_n=Y_1+...+Y_n,$$
$$s_0=t_0=0;\qquad \lim s_n=s;\qquad \lim t_n=t,$$
$$u^{(n)}=X_{\kappa_1}Y_{\lambda_1}+X_{\kappa_2}Y_{\lambda_2}+...+X_{\kappa_n}Y_{\lambda_n}. \tag{11}$$

Then, for some constant C_1,

$$|s_n|,\ |t_n|\leqslant C_1 \tag{12}$$

for all n. Given any real number $\epsilon>0$ there is an m such that

$$|s_mt_m-st|,\ |s_{n_2}-s_{n_1}|,\ |t_{n_2}-t_{n_1}|\leqslant\epsilon \tag{13}$$

whenever $m\leqslant n_1\leqslant n_2$. From the definition of a permutation sequence follows the existence of n_0 such that

$$(\kappa,\lambda)^{(n)}=1\quad(\kappa,\lambda\leqslant m;\ n\geqslant n_0). \tag{14}$$

Now choose any $n\geqslant n_0$. Suppose that κ'_ρ ($\rho\leqslant a$) are all κ with

$$\prod_\lambda\tfrac{1}{2}(1+[\kappa,\lambda]^{(n)})=0,$$

and λ'_σ ($\sigma\leqslant b$) are all λ with

$$\prod_\kappa\tfrac{1}{2}(1+[\kappa,\lambda]'^{(n)})=0.$$

Then
$$a+b=S^{(n)}+S'^{(n)}\leqslant C.$$

Let
$$\{\kappa'_1,\kappa'_2,...,\kappa'_a,0,m\}=\{\kappa''_1,\kappa''_2,...,\kappa''_c\},$$
$$\{\lambda'_1,\lambda'_2,...,\lambda'_a,0,m\}=\{\lambda''_1,\lambda''_2,...,\lambda''_d\},$$

where

$$0 = \kappa''_1 < \kappa''_2 < \ldots < \kappa''_c,$$
$$0 = \lambda''_1 < \lambda''_2 < \ldots < \lambda''_d.$$

Then $$c+d \leqslant a+b+4 \leqslant C+4. \tag{15}$$

One of the $\kappa''_{\mu+1}$ $(\mu < c)$ equals m, say

$$\kappa''_{\mu_0+1} = m.$$

For $$\kappa''_1 = 0 \neq m.$$

Similarly, $$\lambda''_{\nu_0+1} = m$$

for some $\nu_0 < d$. If κ differs from every κ''_ν, and λ is arbitrary, then $(\kappa, \lambda)^{(n)} = (\kappa+1, \lambda)^{(n)}$. Therefore, as in (10),

$$\begin{aligned} u^{(n)} &= \sum_{\kappa,\lambda} \tfrac{1}{2}\big(1+(\kappa,\lambda)^{(n)}\big) X_\kappa Y_\lambda \\ &= \sum_{\substack{\mu<c \\ \nu<d}} \sum_{\substack{\kappa''_\mu < \kappa \leqslant \kappa''_{\mu+1} \\ \lambda''_\nu < \lambda \leqslant \lambda''_{\nu+1}}} \tfrac{1}{2}\big(1+(\kappa''_{\mu+1}, \lambda''_{\nu+1})^{(n)}\big) X_\kappa Y_\lambda \\ &= \sum_{\substack{\mu<c \\ \nu<d}} \tfrac{1}{2}\big(1+(\kappa''_{\mu+1}, \lambda''_{\nu+1})^{(n)}\big)(s_{\kappa''_{\mu+1}} - s_{\kappa''_\mu})(t_{\lambda''_{\nu+1}} - t_{\lambda''_\nu}) \qquad (16) \\ &= \sum_{\substack{\mu<c \\ \nu<d}} u_{\mu\nu}, \quad \text{say.} \end{aligned}$$

In view of (14) we conclude further that

$$u^{(n)} = \sum_{\substack{\mu \leqslant \mu_0 \\ \nu \leqslant \nu_0}} (s_{\kappa''_{\mu+1}} - s_{\kappa''_\mu})(t_{\lambda''_{\nu+1}} - t_{\lambda''_\nu}) + \left(\sum_{\substack{\mu \leqslant \mu_0 \\ \nu_0 < \nu < d}} + \sum_{\substack{\mu_0 < \mu < c \\ \nu < d}} \right) u_{\mu\nu}$$

$$= s_m t_m + \left(\sum_{\substack{\mu \leqslant \mu_0 \\ \nu_0 < \nu < d}} + \sum_{\substack{\mu_0 < \mu < c \\ \nu < d}} \right) u_{\mu\nu}.$$

Therefore, by (12), (13), (15),

$$\begin{aligned} |u^{(n)} - st| &\leqslant |s_m t_m - st| + \mu_0(d-\nu_0-1) \times 2C_1\epsilon + (c-\mu_0-1)(d-1)\epsilon \times 2C_1 \\ &\leqslant \epsilon + 4cdC_1\epsilon \leqslant \epsilon + 4(C+4)^2 C_1 \epsilon. \end{aligned}$$

Since ϵ was arbitrary, and the last inequality holds for any $n \geqslant n_0 = n_0(\epsilon)$, (1) follows, i.e. the permutation sequence $(\kappa_\nu, \lambda_\nu)$ is distributive.

For a later application we note that, if instead of supposing that the sequences s_n, t_n are convergent, we merely suppose that they are bounded, say that

$$|s_n| \leqslant K; \qquad |t_n| \leqslant K'$$

for all n, then (16)—where now no number m need be included among the κ''_μ, λ''_ν and accordingly $c+d \leqslant C+2$—shows that, for every n,

$$|u^{(n)}| \leqslant (c-1)(d-1)(2K)(2K') \leqslant 4C^2KK'.$$

3. Before we start to prove the necessity of (9) it is convenient to set out two lemmas.

LEMMA 1. *Suppose the complex numbers z_ν satisfy the following condition. For some sequence m_ρ, where $m_1 < m_2 < \ldots$, we have, for every ρ,*

$$\Big|\sum_{m_\rho < \nu \leqslant M} z_\nu\Big| \leqslant m_\rho^{-2} \quad (m_\rho < M \leqslant m_{\rho+1}).$$

Then $\sum z_\nu$ is convergent.

Proof. Given any real number $\epsilon > 0$ there is σ such that

$$\sum_{\nu > m_\sigma} \nu^{-2} \leqslant \epsilon.$$

Then, for any $M > m_\sigma$, we have

$$m_\tau < M \leqslant m_{\tau+1}$$

with a suitable $\tau \geqslant \sigma$, and therefore

$$\begin{aligned}\Big|\sum_{m_\sigma < \nu \leqslant M} z_\nu\Big| &\leqslant \sum_{\sigma \leqslant \rho < \tau} \Big|\sum_{m_\rho < \nu \leqslant m_{\rho+1}} z_\nu\Big| + \Big|\sum_{m_\tau < \nu \leqslant M} z_\nu\Big| \\ &\leqslant \sum_{\sigma \leqslant \rho \leqslant \tau} m_\rho^{-2} \leqslant \sum_{\nu > m_\sigma} \nu^{-2} \leqslant \epsilon.\end{aligned}$$

Hence the conclusion.

In what follows $f(k, l, m)$ is the function mentioned in the introduction, in the enunciation of Ramsay's theorem.

LEMMA 2.† *Given any system of*

$$c \geqslant f(2^l, l, 2l-1) = F(l)$$

sets $M_1, M_2, \ldots, M_c$, it is always possible to select l sets

$$M_{a_\lambda} = M'_\lambda \quad (\lambda \leqslant l),$$

where

$$a_1 < a_2 < \ldots < a_l \leqslant c,$$

and to find a number $p \leqslant l+1$ such that, however we choose $r \leqslant l$ numbers e_ρ ($e_1 < e_2 < \ldots < e_r \leqslant l$), we have

$$M'_{e_1} M'_{e_2} \ldots M'_{e_r} \begin{cases} \neq 0 & \text{if } r < p, \\ = 0 & \text{if } p \leqslant r \leqslant l. \end{cases}$$

Proof. Divide all $\binom{c}{l}$ systems $(x_1, x_2, \ldots, x_l)$ $(x_1 < x_2 < \ldots < x_l \leqslant c)$ into classes according to the rule: two such systems $(x'_1, \ldots, x'_l)$, $(x''_1, \ldots, x''_l)$ belong to the same class if and only if, for every $\lambda \leqslant l$,

† This lemma is a special case of a more general proposition which follows easily from Ramsay's theorem.

$$\left.\begin{array}{ll} \text{either} & M_{x_1'} M_{x_2'} \ldots M_{x_\lambda'} = M_{x_1''} M_{x_2''} \ldots M_{x_\lambda''} = 0 \\ \text{or} & M_{x_1'} \ldots M_{x_\lambda'} \neq 0; \qquad M_{x_1''} \ldots M_{x_\lambda''} \neq 0 \end{array}\right\}. \tag{17}$$

The number of classes is at most 2^l. According to the definition of $f(2^l, l, 2l-1)$, there are $2l-1$ numbers a_μ with

$$a_1 < a_2 < \ldots < a_{2l-1} \leqq F(l) \leqq c$$

which have the property that all systems

$$(a_{e_1}, a_{e_2}, \ldots, a_{e_l}) \quad (e_1 < e_2 < \ldots < e_l \leqq 2l-1)$$

belong to the same class. In particular, if

$$r \leqq l; \qquad e_1 < e_2 < \ldots < e_r \leqq l; \qquad f_1 < f_2 < \ldots < f_r \leqq l,$$

then the systems

$$(a_{e_1}, a_{e_2}, \ldots, a_{e_r}, a_{l+1}, a_{l+2}, \ldots, a_{l+r-1}),$$

$$(a_{f_1}, a_{f_2}, \ldots, a_{f_r}, a_{l+1}, a_{l+2}, \ldots, a_{l+r-1})$$

belong to the same class. Hence, putting

$$M_{a_\nu} = M'_\nu \quad (\nu \leqq l)$$

and using (17) with $\lambda = r$, we have

$$\left.\begin{array}{ll} \text{either} & M'_{e_1} M'_{e_2} \ldots M'_{e_r} = M'_{f_1} \ldots M'_{f_r} = 0 \\ \text{or} & M'_{e_1} \ldots M'_{e_r} \neq 0; \qquad M'_{f_1} \ldots M'_{f_r} \neq 0. \end{array}\right.$$

This obviously proves the lemma.

4. We now prove the necessity of the condition (9). We assume that, for a given permutation sequence $(\kappa_\nu, \lambda_\nu)$, (9) is not true, i.e. that

$$\overline{\lim_{n\to\infty}}(S^{(n)} + S'^{(n)}) = \infty.$$

$S^{(n)}$ and $S'^{(n)}$ were defined in § 1. We have to define complex numbers X_κ, Y_λ for which the series $\sum X_\kappa$, $\sum Y_\lambda$ are convergent while the series $\sum_\nu X_{\kappa_\nu} Y_{\lambda_\nu}$ is not convergent. Actually, our definition will be such that for a certain sequence $n' \to \infty$ we have $|u^{(n')}| \to \infty$ while at the same time the numbers

$$\arg(u^{(n')}) = \frac{u^{(n')}}{|u^{(n')}|}$$

are everywhere dense on the unit circle. (We put $\arg(0) = 0$.) $u^{(n)}$ is defined by (11).

Without loss of generality we may assume that

$$\overline{\lim_{n\to\infty}} S^{(n)} = \infty. \tag{18}$$

Case α. Assume that, for some λ_0,

$$\overline{\lim_{n\to\infty}} \sum_{\kappa} \tfrac{1}{2}(1-[\kappa,\lambda_0]^{(n)}) = \infty. \tag{19}$$

We put $$Y_\lambda = \begin{cases} 1 & (\lambda = \lambda_0), \\ 0 & (\lambda \neq \lambda_0). \end{cases}$$

Then $\sum Y_\lambda$ converges. We define the X_κ by induction. Put $X_1 = 1$. Suppose that, for some m, the complex numbers $X_1, X_2, \ldots, X_m$ have already been defined. Choose a complex number ϑ with $|\vartheta| = 1$, and a number n with

$$m^{(n)} > m, \tag{20}$$

and put

$$X_\kappa = (\kappa,\lambda_0)^{(n)} \tfrac{1}{2}(1-[\kappa,\lambda_0]^{(n)}) m^{-2}\vartheta \quad (m < \kappa \leqslant m^{(n)}). \tag{21}$$

By (8), we have

$$\Big|\sum_{m<\kappa\leqslant M} X_\kappa\Big| = |\pm m^{-2}\vartheta(1-1+-\ldots\pm 1)| \leqslant m^{-2} \quad (m < M \leqslant m^{(n)}).$$

We repeat the process of defining the X_κ by equations (21) indefinitely, employing for each step any suitable n, i.e. a number n which satisfies (20) for the m in question, and arbitrary numbers ϑ. We thus obtain, by Lemma 1, a convergent series $\sum X_\kappa$. Furthermore,

$$\begin{aligned} u^{(n)} &= \sum_{\kappa,\lambda\leqslant m^{(n)}} \tfrac{1}{2}\big(1+(\kappa,\lambda)^{(n)}\big) X_\kappa Y_\lambda \\ &= \sum_{\kappa\leqslant m} \tfrac{1}{2}\big(1+(\kappa,\lambda_0)^{(n)}\big) X_\kappa + \\ &\qquad + \sum_{m<\kappa\leqslant m^{(n)}} \tfrac{1}{2}\big(1+(\kappa,\lambda_0)^{(n)}\big)(\kappa,\lambda_0)^{(n)} \tfrac{1}{2}(1-[\kappa,\lambda_0]^{(n)}) m^{-2}\vartheta \\ &= S_1 + S_2, \quad \text{say.} \end{aligned}$$

$$|S_1| \leqslant \sum_{\kappa\leqslant m} |X_\kappa| = S_3; \qquad S_2 = m^{-2}\vartheta Q,$$

$$Q = \sum{}' \tfrac{1}{2}(1-[\kappa,\lambda_0]^{(n)}). \tag{22}$$

$\sum'$ means summation over all κ satisfying

$$m < \kappa \leqslant m^{(n)}, \qquad (\kappa,\lambda_0)^{(n)} = 1. \tag{23}$$

Now (22) shows that Q is equal to the number of κ's which satisfy $[\kappa,\lambda_0]^{(n)} = -1$ as well as (23). Therefore, by (8) and (7),

$$Q \geqslant \tfrac{1}{2} \sum_{m<\kappa\leqslant m^{(n)}} \tfrac{1}{2}(1-[\kappa,\lambda_0]^{(n)}) - 1 \geqslant \tfrac{1}{2} \sum_{\kappa} \tfrac{1}{2}(1-[\kappa,\lambda_0]^{(n)}) - \tfrac{1}{2}m - 1.$$

In view of (19) we can make Q as large as we please by choosing a suitable n. We fix n such that

$$m^{-2}Q \geqslant mS_3.$$

Then

$$|u^{(n)}| \geqslant -|S_1|+|S_2| \geqslant -S_3+mS_3 \geqslant m-1,$$

$$\arg(u^{(n)}\vartheta^{-1}) = \arg(S_1 m^2\vartheta^{-1}Q^{-1}+1) = 1+O(m^{-1}),$$

uniformly in ϑ. If at the various stages of our inductive definition we choose numbers ϑ which form a sequence everywhere dense on the unit circle, we obtain numbers X_κ, Y_λ which have all the properties required.

Case β. Suppose that, for every λ,

$$\max_n \sum_\kappa \tfrac{1}{2}(1-[\kappa,\lambda]^{(n)}) = A_\lambda$$

is finite ($\geqslant 0$), and that

$$\overline{\lim_{\lambda\to\infty}} A_\lambda = \infty. \tag{24}$$

In this case we define both sequences X_κ, Y_λ by induction. Put $X_1 = Y_1 = 1$. Suppose that, for some m, the complex numbers X_1, $X_2,\ldots, X_m$, $Y_1,\ldots, Y_m$ have already been defined. Choose a number $\lambda_0 > m$ for which $A_{\lambda_0} > 0$. Let n be the least number satisfying

$$\sum_\kappa \tfrac{1}{2}(1-[\kappa,\lambda_0]^{(n)}) = A_{\lambda_0}.$$

Then $\lambda_0 < m^{(n)}$. For, if $\lambda \geqslant m^{(n)}$, then $A_\lambda = 0 \neq A_{\lambda_0}$. Put

$$Y_\lambda = \begin{cases} m^{-2} & (\lambda = \lambda_0), \\ 0 & (m < \lambda \leqslant m^{(n)};\ \lambda \neq \lambda_0), \end{cases}$$

$$X_\kappa = (\kappa,\lambda_0)^{(n)}\tfrac{1}{2}(1-[\kappa,\lambda_0]^{(n)})m^{-2}\vartheta \quad (m < \kappa \leqslant m^{(n)}).$$

Here ϑ has the same meaning as above. For the same reason as in Case α the series $\sum X_\kappa$ is convergent. $\sum Y_\lambda$ converges because

$$\Big|\sum_{m<\lambda\leqslant M} Y_\lambda\Big| \leqslant m^{-2} \quad (m < M \leqslant m^{(n)}).$$

Also,

$$\begin{aligned} u^{(n)} &= \sum_{\kappa,\lambda\leqslant m} \tfrac{1}{2}(1+(\kappa,\lambda)^{(n)})X_\kappa Y_\lambda + \\ &\quad + \sum_{m<\kappa\leqslant m^{(n)}} \tfrac{1}{2}(1+(\kappa,\lambda_0)^{(n)})(\kappa,\lambda_0)^{(n)}\tfrac{1}{2}(1-[\kappa,\lambda_0]^{(n)})m^{-2}\vartheta m^{-2} \\ &= S_4+m^{-4}\vartheta Q, \end{aligned}$$

where Q is again given by (22). We have

$$|S_4| \leqslant \sum_{\kappa,\lambda\leqslant m} |X_\kappa Y_\lambda| = S_5.$$

Arguing as before, this time using (24), we see that we can make Q arbitrarily large by choosing λ_0 suitably. Make in particular

$$m^{-4}Q \geqslant mS_5.$$

Then
$$|u^{(n)}| \geqslant -S_5+mS_5 \geqslant m-1,$$
$$\arg(u^{(n)}\vartheta^{-1}) = 1+O(m^{-1}),$$
and the conclusion follows as in case α.

Case γ. Suppose that, for some constant a and every λ, n,
$$\sum_{\kappa} \tfrac{1}{2}(1-[\kappa,\lambda]^{(n)}) < a. \tag{25}$$
Consider, for some fixed n, the sets
$$A_\kappa = \sum_{\lambda} \{\tfrac{1}{2}(1-[\kappa,\lambda]^{(n)})\lambda\}-\{0\}.$$
Clearly,
$$1-\prod_{\lambda} \tfrac{1}{2}(1+[\kappa,\lambda]^{(n)}) = \begin{cases} 1 & \text{if } A_\kappa \neq 0, \\ 0 & \text{if } A_\kappa = 0. \end{cases} \tag{26}$$
Furthermore,
$$A_{\nu_1}A_{\nu_2}\ldots A_{\nu_a} = 0 \quad (\nu_1 < \nu_2 < \ldots < \nu_a). \tag{27}$$
For
$$\lambda' \text{ in } A_{\nu_\rho} \quad (\rho \leqslant a)$$
would imply that
$$[\nu_\rho,\lambda']^{(n)} = -1 \quad (\rho \leqslant a),$$
$$\sum_{\kappa} \tfrac{1}{2}(1-[\kappa,\lambda']^{(n)}) \geqslant \sum_{\rho\leqslant a} \tfrac{1}{2}(1-[\nu_\rho,\lambda']^{(n)}) = a,$$
which contradicts (25).

Put $X_1 = Y_1 = 1$. Suppose that, for some m, the complex numbers $X_1,\ldots, X_m, Y_1,\ldots, Y_m$ have already been defined. Choose a number n which will be specified later. Denote by
$$\beta_1, \beta_2,\ldots, \beta_c$$
all numbers $\kappa > m$ for which $A_\kappa \neq 0$, arranged so that
$$m < \beta_1 < \beta_2 < \ldots < \beta_c.$$
Then, by (26), (7),
$$m < \beta_1 < \beta_2 < \ldots < \beta_c < m^{(n)}.$$
Moreover,
$$c = \sum_{\kappa>m}\Big(1-\prod_{\lambda} \tfrac{1}{2}(1+[\kappa,\lambda]^{(n)})\Big) \geqslant \sum_{\kappa}\Big(1-\prod_{\lambda} \tfrac{1}{2}(1+[\kappa,\lambda]^{(n)})\Big)-m$$
$$= S^{(n)}-m.$$
Therefore, by (18), we can make c arbitrarily large by choosing n appropriately. Put
$$M_\nu = A_{\beta_\nu} \quad (\nu \leqslant c).$$
Then $M_\nu \neq 0$. Choose a number $l \geqslant a$ and make $c \geqslant F(l)$ where

$F(l)$ is the number mentioned in Lemma 2. Then, by Lemma 2, there are l numbers a_λ with

$$a_1 < a_2 < \ldots < a_l \leqslant c$$

and a number $p \leqslant l+1$ such that, putting

$$M'_\nu = M_{a_\nu} \quad (\nu \leqslant l),$$

we have
$$M'_{e_1} M'_{e_2} \ldots M'_{e_r} \begin{cases} \neq 0 & \text{if } r < p, \\ = 0 & \text{if } p \leqslant r \leqslant l. \end{cases}$$

Here the e_ρ may be arbitrary numbers satisfying

$$e_1 < e_2 < \ldots < e_r \leqslant l.$$

Since
$$M'_1 \neq 0$$

and, by (27),
$$M'_1 M'_2 \ldots M'_a = 0,$$

we have
$$1 < p \leqslant a.$$

Put
$$M^* = \begin{cases} M'_1 M'_2 \ldots M'_{p-2} & (p > 2), \\ \{1, 2, 3, \ldots\} & (p = 2). \end{cases}$$

Then
$$M^* M'_\nu \neq 0 \quad (p-1 \leqslant \nu \leqslant l),$$
$$M^* M'_\nu M'_\mu = 0 \quad (p-1 \leqslant \nu < \mu \leqslant l).$$

Choose numbers

$$\lambda'_\nu \text{ in } M^* M'_{p+2\nu} \quad (\nu \leqslant g),$$

where
$$g = \tfrac{1}{2}(l-p) \geqslant \tfrac{1}{2}(l-a).$$

By making l sufficiently large we can make g arbitrarily large. Then

$$\lambda'_\nu \text{ in } M'_{p+2\nu} = A_{\kappa'_\nu}, \tag{28}$$

where
$$\kappa'_\nu = \beta_{a_{p+2\nu}} \quad (\nu \leqslant g).$$

Also, if $\mu, \nu \leqslant g$; $\mu \neq \nu$,

$$\lambda'_\nu \text{ in } M^* M'_{p+2\nu}; \qquad \lambda'_\nu \text{ not in } M^* M'_{p+2\nu} M'_{p+2\mu},$$
$$\lambda'_\nu \text{ not in } M'_{p+2\mu} = A_{\kappa'_\mu} \quad (\mu \neq \nu;\ \mu, \nu \leqslant g). \tag{29}$$

According to the definition of A_κ, (28) and (29) are the same as

$$[\kappa'_\mu, \lambda'_\nu]^{(n)} = \begin{cases} -1 & (\mu = \nu \leqslant g), \\ 1 & (\mu \neq \nu;\ \mu, \nu \leqslant g). \end{cases}$$

In particular, it follows that the λ'_ν are different from each other. Also, $m < \kappa'_\nu$ and
$$\kappa'_\nu + 1 < \kappa'_{\nu+1} \quad (\nu < g).$$

Let δ_ρ ($\rho \leqslant h$) be those numbers $\nu \leqslant g$ for which $\lambda'_\nu > m$. Order the δ_ρ so that
$$\delta_1 < \delta_2 < \ldots < \delta_h \leqslant g.$$

We have $h \geqslant g-m$. Hence h can be made as large as we please. Put

$$i_\nu = \kappa'_\delta\,; \qquad j_\nu = \lambda'_{\delta_\nu} \quad (\nu \leqslant h).$$

Then
$$[i_\mu, j_\nu]^{(n)} = \begin{cases} -1 & (\mu = \nu \leqslant h), \\ 1 & (\mu \neq \nu;\ \mu, \nu \leqslant h). \end{cases} \tag{30}$$

The j_ν are different from each other. Furthermore,

$$m < i_\nu, j_\nu \quad (\nu \leqslant h),$$
$$i_\nu + 1 < i_{\nu+1} \quad (\nu < h). \tag{31}$$

The first line of (30) in conjunction with (7) shows that

$$i_\nu, j_\nu < m^{(n)} \quad (\nu \leqslant h).$$

For any $\nu \leqslant h$, denote by σ_ν the number of indices μ for which

$$\mu \leqslant h; \qquad j_\mu \leqslant j_\nu.$$

Our inductive definition of X_κ, Y_λ is

$$Y_{j_\nu} = (-1)^{\sigma_\nu} m^{-2} \quad (\nu \leqslant h),$$
$$Y_\lambda = 0 \qquad\qquad (m < \lambda \leqslant m^{(n)};\ \lambda \neq j_1, j_2, \ldots, j_h),$$
$$X_{i_\nu} Y_{j_\nu} (i_\nu, j_\nu)^{(n)} = X_{i_\nu+1} Y_{j_\nu} (i_\nu+1, j_\nu)^{(n)} = \vartheta m^{-4} \quad (\nu \leqslant h), \tag{32}$$
$$X_\kappa = 0 \quad (m < \kappa \leqslant m^{(n)};\ \kappa \neq i_1, i_2, \ldots, i_h, i_1+1, \ldots, i_h+1).$$

In view of (31) these equations are consistent. Here ϑ has the same meaning as before. We have, by (30),

$$X_{i_\nu} = \pm\vartheta m^{-2}, \qquad X_{i_\nu} + X_{i_\nu+1} = 0 \quad (\nu \leqslant h). \tag{33}$$

Hence, if $m < M \leqslant m^{(n)}$,

$$|X_{m+1} + \ldots + X_M| = |\pm\vartheta m^{-2}(1 - 1 + - \ldots \pm 1)| \leqslant m^{-2}.$$

Furthermore, we deduce from the definition of σ_ν that

$$|Y_{m+1} + \ldots + Y_M| = |m^{-2}(-1+1-+\ldots\pm 1)| \leqslant m^{-2}.$$

Therefore $\sum X_\kappa$, $\sum Y_\lambda$ are convergent, by Lemma 1. On the other hand,

$$\begin{aligned} w^{(n)} &= \sum_{\kappa,\lambda} \tfrac{1}{2}(1+(\kappa,\lambda)^{(n)}) X_\kappa Y_\lambda \\ &= \sum_{\kappa,\lambda \leqslant m} \tfrac{1}{2}(1+(\kappa,\lambda)^{(n)}) X_\kappa Y_\lambda + \\ &\quad + \sum_{\mu,\nu \leqslant h} \left(\tfrac{1}{2}(1+(i_\mu, j_\nu)^{(n)}) X_{i_\mu} Y_{j_\nu} + \tfrac{1}{2}(1+(i_\mu+1, j_\nu)^{(n)}) X_{i_\mu+1} Y_{j_\nu}\right) \\ &= S_6 + S_7, \quad \text{say.} \end{aligned}$$

$$|S_6| \leqslant \sum_{\kappa,\lambda \leqslant m} |X_\kappa Y_\lambda| = S_8.$$

In view of (33), (30), (32),

$$S_7 = \sum_{\mu,\nu\leqslant h} \tfrac{1}{2}((i_\mu, j_\nu)^{(n)} - (i_\mu+1, j_\nu)^{(n)}) X_{i_\mu} Y_{j_\nu}$$
$$= \sum_{\mu,\nu\leqslant h} \tfrac{1}{2}(1-[i_\mu, j_\nu]^{(n)})(i_\mu, j_\nu)^{(n)} X_{i_\mu} Y_{j_\nu}$$
$$= \sum_{\nu\leqslant h} (i_\nu, j_\nu)^{(n)} X_{i_\nu} Y_{j_\nu} = \vartheta m^{-4} h.$$

By choosing n suitably we can make h as large as we please. Make

$$m^{-4}h > mS_8.$$

Then
$$|u^{(n)}| \geqslant -|S_6| + |S_7| \geqslant -S_8 + mS_8 \geqslant m-1,$$
$$\arg(u^{(n)}\vartheta^{-1}) = 1 + O(m^{-1}),$$

and the construction is completed as in Case α.

5. The series $\sum X_\kappa$, $\sum Y_\lambda$ constructed in § 4 are convergent and have therefore, *a fortiori*, bounded partial sums. The partial sums of the corresponding series $\sum_\nu X_{\kappa_\nu} Y_{\lambda_\nu}$ are not bounded. If we compare this fact with the remark at the end of § 2 we obtain the following result.

THEOREM B. *Distributive permutation sequences $(\kappa_\nu, \lambda_\nu)$ are characterized by the following property. Whenever the partial sums of two series $\sum X_\kappa$, $\sum Y_\lambda$ are bounded, then the partial sums of $\sum_\nu X_{\kappa_\nu} Y_{\lambda_\nu}$ are bounded. In fact, for distributive permutation sequences,*

$$\max_n \Big|\sum_{\nu\leqslant n} X_{\kappa_\nu} Y_{\lambda_\nu}\Big| \leqslant B \max_k \Big|\sum_{\kappa\leqslant k} X_\kappa\Big| \max_l \Big|\sum_{\lambda\leqslant l} Y_\lambda\Big|,$$

where the constant B depends on the numbers κ_ν, λ_ν only.

A THEOREM ON HAUSDORFF MEASURE

By E. BEST (*Belfast*)

[Received 3 May 1940]

1. THE object of this paper is to prove a general theorem concerning the Hausdorff dimension-function of sets of points of a certain type. I give first a brief outline of the essential details of Hausdorff's† theory.

Suppose $h(t)$ is any function of the real variable t such that $h(t)$ is continuous, concave, and strictly increasing in $0 \leqslant t \leqslant t_0$ with $h(0) = 0$. Let E be any linear set. We cover E with any finite or enumerable set of intervals $I(E, \rho)$, whose lengths a_r are all less than ρ $(\leqslant t_0)$; then we define

$$h\text{-}m^*E = \lim_{\rho \to 0} \text{lower bound} \sum_{I(E,\rho)} h(a_r)$$

to be the *exterior h-measure* of E. We define *measurability* in the usual way and all the usual theorems concerning measurable sets hold. We denote the *h-measure* of E by $h\text{-}mE$.

If and only if $$0 < h\text{-}mE < \infty,$$

we say that $h(t)$ is the *dimension-function* of E.

2. We define a sequence $I_0, I_1, \ldots, I_n, \ldots$ of sets of intervals. I_0 is the closed interval $(0, 1)$. I_n consists of T_n closed intervals each of length d_n, and I_{n+1} is obtained from I_n by removing from each of its constituent intervals $(\theta_{n+1}-1)$ open intervals each of length c_{n+1} and leaving θ_{n+1} closed intervals each of length d_{n+1}. We write

$$J_n = I_0 - I_n.$$

There are two obvious relations:

$$T_{n+1} = \theta_{n+1} T_n, \qquad (\theta_{n+1}-1)c_{n+1} + \theta_{n+1} d_{n+1} = d_n.$$

The set $$E = \lim_{n \to \infty} (I_0 I_1 \ldots I_n)$$

is a closed non-null set. If there is a number N such that $c_{n-1} > c_n$ when $n \geqslant N$, we shall say that E is a set of type P. We prove the following theorem concerning the dimension-functions of such sets.

† F. Hausdorff, 'Dimension und äusseres Mass': *Math. Annalen*, 79 (1919), 157–79.

THEOREM. *If E is a set of type P and*

(i) *$h(t)$ is a concave, continuous, and strictly increasing function of the real variable t for $0 \leqslant t \leqslant t_0$, $h(0) = 0$, and $h(t) \geqslant \alpha > 0$ for $t_0 \leqslant t \leqslant 1$;*

(ii) $$\varliminf_{n\to\infty} T_n h(d_n) = \lambda_1, \qquad \varlimsup_{n\to\infty} T_n h(d_n) = \lambda_2,$$

where $$0 < \lambda_1 \leqslant \lambda_2 < \infty,$$

then $h(t)$ is the dimension-function of the set E.

From (ii) it follows at once that

$$h\text{-}mE \leqslant \lambda_2 < \infty.$$

3. To prove $h\text{-}mE > 0$ we make use of the following theorem due to Gillis:†

THEOREM. *Given a closed linear set E of finite h-measure in $0 \leqslant x \leqslant 1$, a necessary and sufficient condition that $h\text{-}mE > 0$ is that there should exist a closed subset E_1 of E and a function $\phi(x)$ such that*

(i) *$\phi(x)$ is continuous and non-decreasing in $(0, 1)$;*

(ii) *$\phi(x)$ is constant in each contiguous interval of E_1;*

(iii) *as x describes E_1, $\phi(x)$ describes a set of positive Lebesgue measure;*

(iv) *if x, x' belong to E_1, then $|\phi(x)-\phi(x')| \leqslant \mu h(|x-x'|)$, where μ is a constant.*

We define a function $f_n(x)$ as follows: $f_n(0) = 0$, $f_n(1) = 1$, and, in each interval of I_n, $f_n(x)$ is linear and continuous, having an increment of $1/T_n$ in that interval; in each interval of J_n, $f_n(x)$ is constant and the constants are so adjusted that the function is everywhere continuous; it will also be monotonic.

If ξ_1 and ξ_2 are the end-points of any interval of length d_n, then

$$f_n(\xi_1) = f_{n+r}(\xi_1), \qquad f_n(\xi_2) = f_{n+r}(\xi_2) \quad (r = 1, 2, \ldots);$$

and $f_n(x)$ converges uniformly to a function $f(x)$. Since the conditions (i), (ii), (iii) of the theorem are obviously true for each $f_n(x)$, they are therefore true for $f(x)$.‡

† J. Gillis, 'Note on a theorem of Myrberg': *Proc. Cambridge Phil. Soc.* 33 (1937), 419–24.

‡ In particular (iii) follows because $f(1)-f(0) = 1$ and because $f(x)$ is constant in every contiguous interval.

We shall prove the inequality

$$|f_n(x)-f_n(x')| \leqslant \mu h(|x-x'|), \tag{1}$$

where x, x' lie in I_n and μ is a constant.

If $|x-x'| \geqslant t_0$, then, by (i), $h(|x-x'|) \geqslant \alpha$, and $\mu = \mu_1$ can be selected so that the inequality is satisfied, since the left-hand side never exceeds unity. In what follows we therefore assume that $|x-x'| < t_0$.

The least favourable case for the inequality will always occur when, if x, x' belong to different intervals of I_n, they are at the ends of those intervals which are farthest apart and, if they belong to the same interval, they are at its ends. We consider then, first, the case when x, x' belong to the same interval of I_n. We have to prove

$$|f_n(x)-f_n(x')| = 1/T_n \leqslant \mu h(d_n).$$

Now $\underline{\lim}_{n\to\infty} T_n h(d_n) = \lambda_1$; hence

$$T_n h(d_n) \geqslant \lambda_1/\nu \quad (n \geqslant n_0),$$

where ν is a constant; therefore the inequality will be satisfied for $\mu \geqslant \nu/\lambda_1$ and $n \geqslant n_0$.

We now consider the case where x and x' belong to the same interval of I_{n-1} but to different intervals of I_n. Let the number of intervals of I_n between the intervals of I_n to which x and x' belong be $r-2$. As we have just shown, for $\mu \geqslant \nu/\lambda_1$ and $n-1 \geqslant n_0$, we have

$$\frac{1}{T_{n-1}\theta_n} = \frac{1}{T_n} \leqslant \mu h(d_n),$$

$$\frac{\theta_n}{T_{n-1}\theta_n} = \frac{1}{T_{n-1}} \leqslant \mu h(d_{n-1}) = \mu h\left[d_n + \frac{d_{n-1}-d_n}{\theta_n - 1}(\theta_n - 1)\right],$$

and we wish to prove

$$\frac{r}{T_{n-1}\theta_n} = |f_n(x)-f_n(x')| \leqslant \mu h(|x-x'|) = \mu h\left[d_n + \frac{d_{n-1}-d_n}{\theta_n - 1}(r-1)\right].$$

This follows at once from the concavity of $h(t)$.

We have now to consider the case where x, x' belong to different intervals of both I_n and I_{n-1}. We suppose that the inequality (1) is true for suitable μ (at least greater than ν/λ_1) and $n = k$, and we will show that it is true for the same μ and $n = k+1$.

Let AB, $A'B'$ be two intervals belonging to I_k (A, B, A', B' being the end-points), AB being the interval nearer the origin; let

$A''B''$ be the nearest interval of I_k which lies just to the left of the interval $A'B'$ (B'' and B may coincide). Then the inequality is satisfied for x at A and x' at both B and B'. Let CD and $C'D'$ be two intervals of I_{k+1} lying within AB and $A'B'$ respectively; further, let CD be the rth interval and $C'D'$ be the qth interval of I_{k+1} counting from A and A' respectively. Then we have to prove the inequality (1) to be true for x at C and x' at D'. There are two cases to be considered.

(i) $q \geqslant r$. We take x_1 at A, x_1'' at A'', x_2'' at B'', x_2' at B', ξ at C, and η at D'.

There is no loss of generality in supposing that C coincides with A and that $C'D'$ is moved a corresponding distance to the left, since both $|f_{k+1}(\zeta)-f_{k+1}(\eta)|$ and $|\eta-\zeta|$ are unchanged by the process. We have, by hypothesis,

$$|f_{k+1}(x_2'')-f_{k+1}(x_1)| = |f_k(x_2'')-f_k(x_1)| < \mu h(|x_2''-x_1|),$$

$$|f_{k+1}(x_2')-f_{k+1}(x_1)| = |f_k(x_2')-f_k(x_1)| < \mu h(|x_2'-x_1|).$$

Also, since $B''A' \geqslant c_k$, it is easily seen by elementary geometry that

$$\frac{f_{k+1}(x_2')-f_{k+1}(x_2'')}{x_2'-x_2''} \geqslant \frac{f_{k+1}(\eta)-f_{k+1}(x_2'')}{\eta-x_2''}.$$

It follows immediately from the concavity of $h(t)$ that

$$|f(\eta)-f(\zeta)| \leqslant \mu h(|\zeta-\eta|).$$

(ii) $q < r$. We have

$$CD' = |\zeta-\eta| = |x_1-x_1''|+B''A'+d_k+(q-r)(c_{k+1}+d_{k+1})+d_{k+1}$$

and

$$|f_{k+1}(\xi)-f_{k+1}(\eta)| = |f_{k+1}(x_1)-f_{k+1}(x_1'')|+\frac{\theta_{k+1}+q-r+1}{T_{n+1}}.$$

We consider three cases.

(a) $CD' \geqslant AB''$. The inequality (1) is satisfied for x_1 at A and x_2'' at B''. Hence in replacing x_1 by ξ and x_2'' by η we do not decrease the right-hand side of the inequality nor do we increase the left-hand side since $\theta_{k+1}+q-r+1 \leqslant \theta_{k+1}$. Hence the inequality will be satisfied in this case also.

(b) $CD' < AB''$, and B'' not coincident with B. By what we have just proved in (i), if ξ' is the left-hand end-point of the first interval of I_{k+1} in AB and η' is the right-hand end-point of the

$(\theta_{k+1}+q-r+1)$th interval of I_{k+1} in $A''B''$, then the inequality is true. But

$$
\begin{aligned}
&|\xi'-\eta'| \\
&= |x_1-x_1''|+(\theta_{k+1}+q-r)(c_{k+1}+d_{k+1})+d_{k+1} \\
&= |x_1-x_1''|+\theta_{k+1}d_{k+1}+(\theta_{k+1}-1)c_{k+1}+c_{k+1}+(q-r)(c_{k+1}+d_{k+1})+d_{k+1} \\
&= |x_1-x_1''|+d_k+c_{k+1}+(q-r)(c_{k+1}+d_{k+1})+d_{k+1} \\
&\leqslant |x_1-x_1''|+B''A'+d_k+(q-r)(c_{k+1}+d_{k+1})+d_{k+1} \\
&= |\zeta-\eta|,
\end{aligned}
$$

since $B''A' \geqslant c_k \geqslant c_{k+1}$. But obviously

$$|f_{k+1}(\xi')-f_{k+1}(\eta)| = |f_{k+1}(\xi)-f_{k+1}(\eta)|.$$

Thus the inequality is true in this case also.

(c) $CD' < AB''$, and B'' coincident with B. This follows in an exactly similar fashion to (b) except that instead of using case (i) we use the fact that the inequality is true when we have two intervals of length I_{k+1} arising from the same interval of I_k, in which we have already proved the inequality (1) to be true.

Hence the inequality (1) is true for $n = k+1$ if it is true for $n = k$ $(k \geqslant n_0)$ with $\mu = \mu_1$, say. When $k = n_0$ we can always pick a finite $\mu = \mu_2$ to make the inequality (1) true, and therefore it is true in general for $n \geqslant k_0$,

i.e. $$|f_n(\xi)-f_n(\eta)| \leqslant \mu_3 h(|\eta-\xi|),$$

where $\mu_3 = \max(\mu_1, \mu_2)$.

Considering now also the first two cases which concerned single intervals of I_k and I_{k+1} we have, if $\mu = \mu_0 = \max(\mu_3, \nu/\lambda_1)$,

$$|f_n(x)-f_n(x')| \leqslant \mu_0 h(|x-x'|) \quad (n \geqslant n_0),$$

x, x' belonging to I_n.

Proceeding now to the limit we obtain

$$|f(x)-f(x')| \leqslant \mu_0 h(|x-x'|)$$

for all x, x' belonging to E. This gives us the fourth condition of the theorem of Gillis and hence $h\text{-}mE > 0$. Thus $h(t)$ is the dimension-function of the set E.

5. I now proceed to give an application of the theorem to a simple set—that of Cantor. This is obtained by dividing the interval $(0, 1)$ into thirds and removing the inner intervals; we then divide

each outer interval into thirds and remove the inner interval again; and so on.

Obviously this is a set of type P. We take

$$h(t) = t^{\alpha}, \qquad \alpha = (\log 2)/(\log 3)$$

and condition (i) of the theorem is satisfied. Further

$$T_n = 2^n, \qquad d_n = 3^{-n},$$

and therefore $$T_n h(d_n) = 2^n 3^{-n\alpha},$$

i.e. $$\log[T_n h(d_n)] = n \log 2 - n \frac{\log 2}{\log 3} \log 3 = 0,$$

i.e. $$T_n h(d_n) = 1$$

and condition (ii) is satisfied. Hence t^{α} is the dimension-function of the set.†

† I am indebted to the referee for several suggestions which have enabled me to clarify the argument.

EXPANSIONS OF APPELL'S DOUBLE HYPERGEOMETRIC FUNCTIONS

By J. L. BURCHNALL (*Durham*) and T. W. CHAUNDY (*Oxford*)

[Received 26 August 1940]

1. Introduction: symbolic forms

In 1933 Bailey showed* that Appell's double hypergeometric function of the fourth type reduces to the product of two ordinary hypergeometric functions when its parameters are connected by a single relation. Contemplation of this result suggested that, for the unrestricted Appell's function, it might well represent the first term in an infinite series of such products. This conjecture was confirmed on examination, but a number of other expansions of Appell's function were also revealed. These we give here, developing them in what seems to be their natural order: in this the particular expansion with which our inquiry began appears somewhat late.

We recall then, in slightly modified notation,† the four Appell's functions as defined by their expansions:‡

$$F^{(1)}[a;b,b';c;x,y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{m!\,n!\,(c)_{m+n}} x^m y^n, \tag{1}$$

$$F^{(2)}[a;b,b';c,c';x,y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{m!\,n!\,(c)_m(c')_n} x^m y^n, \tag{2}$$

$$F^{(3)}[a,a';b,b';c;x,y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n}{m!\,n!\,(c)_{m+n}} x^m y^n, \tag{3}$$

$$F^{(4)}[a,b;c,c';x,y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{m!\,n!\,(c)_m(c')_n} x^m y^n, \tag{4}$$

where, as usual, $(a)_r \equiv \Gamma(a+r)/\Gamma(a)$. The series $F^{(1)}$, $F^{(3)}$ converge (absolutely) when $|x|$, $|y| < 1$; $F^{(2)}$ when $|x|+|y| < 1$; and $F^{(4)}$ when $|\sqrt{x}|+|\sqrt{y}| < 1$.

* W. N. Bailey, (2) and (3); or (4), 81, § 9.6.

† To avoid confusion with the simple hypergeometric functions ${}_rF_s$.

‡ P. Appell and J. Kampé de Fériet (1), 14 (11)–(14) or Bailey (4), 73 (1)–(4).

We associate with these four definitions the two elementary expansions

$$F(a,b;c;x)F(a',b';c';y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n}{m!\,n!\,(c)_m(c')_n} x^m y^n,$$

$$F(a,b;c;x+y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{m!\,n!\,(c)_{m+n}} x^m y^n,$$

and introduce the inverse pair of symbolic operators

$$\nabla(h) \equiv \frac{\Gamma(h)\Gamma(\delta+\delta'+h)}{\Gamma(\delta+h)\Gamma(\delta'+h)}, \qquad \Delta(h) \equiv \frac{\Gamma(\delta+h)\Gamma(\delta'+h)}{\Gamma(h)\Gamma(\delta+\delta'+h)}, \tag{5}$$

where $\delta,\ \delta' \equiv x\partial/\partial x,\ y\partial/\partial y$.

Then $$\nabla(h)(h)_m(h)_n x^m y^n = (h)_{m+n} x^m y^n,$$

and so, if $(h)_m(h)_n$ occurs in the numerator of the coefficient of $x^m y^n$, it is changed into $(h)_{m+n}$ by the operator $\nabla(h)$. The operator $\Delta(h)$ effects a similar change in the denominator. Now Appell's series can be obtained from the product of two simple hypergeometric series by changes of this character* and we can therefore write symbolically

$$F^{(2)}[a;b,b';c,c';x,y] = \nabla(a)F(a,b;c;x)F(a,b';c';y), \tag{6}$$

$$F^{(3)}[a,a';b,b';c;x,y] = \Delta(c)F(a,b;c;x)F(a',b';c;y), \tag{7}$$

$$F^{(1)}[a;b,b';c;x,y] = \nabla(a)\Delta(c)F(a,b;c;x)F(a,b';c;y), \tag{8}$$

$$F^{(1)}[a;b,b';c;x,y] = \nabla(a)F^{(3)}[a,a;b,b';c;x,y], \tag{9}$$

$$F^{(1)}[a;b,b';c;x,y] = \Delta(c)F^{(2)}[a;b,b';c,c;x,y], \tag{10}$$

$$F^{(4)}[a,b;c,c';x,y] = \nabla(b)F^{(2)}[a;b,b;c,c';x,y], \tag{11}$$

$$F(a,b;c;x+y) = \nabla(b)F^{(1)}[a;b,b;c;x,y], \tag{12}$$

$$F(a,b;c;x+y) = \Delta(c)F^{(4)}[a,b;c,c;x,y], \tag{13}$$

$$F(a,b;c;x+y) = \nabla(b)\Delta(c)F^{(2)}[a;b,b;c,c;x,y]. \tag{14}$$

We note further that on a simple hypergeometric series we have

$$\frac{1}{(\delta+c)_r} F(a,b;c;x) = \frac{1}{(c)_r} F(a,b;c+r;x) \tag{15}$$

with similar formulae for the double series, provided that we replace δ by δ', $\delta+\delta'$ when the parameter c occurs with suffix n, $m+n$ instead of m.

Again, in the coefficient of $x^m y^n$, $1/(-a-\delta-\delta'+1)_r$ changes $(a)_{m+n}$ into

$$(-)^r \frac{(a)_r(a-r)_{m+n}}{(a-r)_{2r}},$$

* Appell and Kampé de Fériet (**1**), 13 or Bailey (**4**), 76.

so that, for instance,

$$\frac{1}{(-a-\delta-\delta'+1)_r} F^{(1)}[a;b,b';c,c';x,y] = (-)^r \frac{(a)_r}{(a-r)_{2r}} F^{(1)}[a-r;b,b';c,c';x,y]. \quad (16)$$

Finally, we note that

$$(-\delta)_r F(a,b;c;x) = (-)^r \frac{(a)_r (b)_r}{(c)_r} x^r F(a+r,b+r;c+r;x), \quad (17)$$

the corresponding results for the double series being sufficiently shown by the formula

$$(-\delta)_r(-\delta')_r F^{(1)}[a;b,b';c;x,y] = \frac{(a)_{2r}(b)_r(b')_r}{(c)_{2r}} x^r y^r F^{(1)}[a+2r;b+r,b'+r;c+2r;x,y], \quad (18)$$

where $2r$ replaces r in association with the 'double' parameters a, c, i.e. parameters which have the suffix $m+n$ in the defining series. Combining (16), (18) we find that

$$\frac{(-\delta)_r(-\delta')_r}{(-a-\delta-\delta'+1)_r} F^{(1)}[a;b,b';c,c';x,y] = (-)^r \frac{(a)_r (b)_r (b')_r}{(c)_{2r}} x^r y^r F^{(1)}[a+r;b+r,b'+r;c+2r;x,y]. \quad (19)$$

2. A set of lemmas

We quote as lemmas the known identities

$$\frac{\Gamma(h)\Gamma(m+n+h)}{\Gamma(m+h)\Gamma(n+h)} = \sum_{r=0}^{\infty} \frac{(-m)_r(-n)_r}{r!\,(h)_r}, \quad (20)$$

$$\frac{\Gamma(m+h)\Gamma(n+h)}{\Gamma(h)\Gamma(m+n+h)} = \sum_{r=0}^{\infty} \frac{(-m)_r(-n)_r}{r!\,(-h-m-n+1)_r} \quad (21)$$

$$= \sum_{r=0}^{\infty} (-)^r \frac{(h)_{2r}(-m)_r(-n)_r}{r!\,(h+r-1)_r(m+h)_r(n+h)_r}, \quad (22)$$

$$\frac{\Gamma(h)\Gamma(m+n+h)\Gamma(m+k)\Gamma(n+k)}{\Gamma(m+h)\Gamma(n+h)\Gamma(k)\Gamma(m+n+k)} = \sum_{r=0}^{\infty} \frac{(k-h)_r(k)_{2r}(-m)_r(-n)_r}{r!\,(k+r-1)_r(m+k)_r(n+k)_r(h)_r} \quad (23)$$

$$= \sum_{r=0}^{\infty} \frac{(h-k)_r(-m)_r(-n)_r}{r!\,(h)_r(-k-m-n+1)_r}. \quad (24)$$

Of these (20) is Gauss's theorem (or Vandermonde's theorem if m, n are positive integers), and (21) is a variant of (20) valid only when one of m, n is an integer; (22), (23) are limiting forms of Dougall's theorem given by Bailey;* and (24) is Saalschütz's theorem,† valid only when one of m, n is a positive integer.

We illustrate our method by first considering a particular example. In (20) replace m, n by δ, δ', so that we have symbolically

$$\nabla(h) = \sum_{r=0}^{\infty} \frac{(-\delta)_r(-\delta')_r}{r!\,(h)_r}.$$

Thus in (6) we have

$$\begin{aligned} &F^{(2)}[a;b,b';c,c';x,y] \\ &\quad = \sum_{r=0}^{\infty} \frac{(-\delta)_r(-\delta')_r}{r!\,(a)_r} F(a,b;c;x)F(a,b';c';y) \\ &\quad = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(b')_r}{r!\,(c)_r(c')_r} x^r y^r F(a+r,b+r;c+r;x)F(a+r,b'+r;c'+r;y) \end{aligned}$$

in virtue of (17), which gives $F^{(2)}$ expanded in a series of products of simple hypergeometric functions. So far the argument has been purely symbolic, but we can make a more convincing use of our material. For write $\mathbf{C}_{m,n}\,S(x,y)$ to denote the coefficient of $x^m y^n$ in a double series $S(x,y)$ with an analogous symbol for the simple series. Then, if the expansion above is to be valid, we must have, on equating corresponding coefficients on the two sides,

$$\begin{aligned} \mathbf{C}_{m,n}\,F^{(2)}[a;b,b';c,c';x,y] &= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(b')_r}{r!\,(c)_r(c')_r} \times \\ &\times \mathbf{C}_{m-r,n-r}\,F(a+r,b+r;c+r;x)F(a+r,b'+r;c'+r;y). \end{aligned} \tag{25}$$

But, from the definition of $F^{(2)}$, or equivalently by (6),

$$\begin{aligned} &\mathbf{C}_{m,n}\,F^{(2)}[a;b,b';c,c';x,y] \\ &\qquad = \frac{\Gamma(a)\Gamma(a+m+n)}{\Gamma(a+m)\Gamma(a+n)} \mathbf{C}_{m,n}\,F(a,b;c;x)F(a,b';c';y); \end{aligned}$$

and we can replace (17) by

$$(-m)_r\,\mathbf{C}_m\,F(a,b;c;x) = (-)^r \frac{(a)_r(b)_r}{(c)_r} \mathbf{C}_{m-r}\,F(a+r,b+r;c+r;x)$$

* Bailey (**4**), 27–8, § 4.4 (3), (1). The first formula, as Bailey points out, is the limit, as $k \to \infty$, of the second, which in this form is given by Dougall (**5**), 122 (9).

† Saalschütz (**6**), 279 (1), quoted in Bailey (**4**), 9, § 2.2 (1).

and the similar formula in y. To complete the proof of (25) we need then only to employ the lemma (20) in m, n as written.

Thus, if in the symbolic analysis we replace δ, δ' by m, n, we can adapt the argument to give term-by-term identification of the two sides of the presumed identity.* This is perhaps obvious from first principles, for, in operation on $x^m y^n$, δ, δ' become replaced by m, n. Now in the term-by-term identification we are, in fact, rearranging the summation of a multiple series. This we justify later by considerations of absolute convergence. For the present we are content with purely formal results, and, as we have just seen, these are adequately established by the symbolic analysis.

Then from the nine identities (6)–(14), if we use the lemmas (20), (22), (23) to express ∇, Δ, $\nabla\Delta$ and appeal to the principles of (15) and (17) or (18), we obtain nine expansions including the expansion for $F^{(2)}$ given above. By 'inverting' these we obtain a further nine. Thus we can rewrite (6) as

$$F(a,b;c;x)F(a,b';c';y) = \Delta(a)F^{(2)}[a;b,b';c,c';x,y],$$

and so on. We must now use the lemmas (21), (24) instead of (22), (23) to give Δ, $\nabla\Delta$, and we shall need to appeal to the principle of (16) or rather (19). These nine inverse expansions lack full generality in the sense that the hypergeometric function to be expanded necessarily has a pair of parameters equal.

We now state these eighteen expansions without further proof, putting each inverse series directly after the corresponding direct series and arranging the nine pairs in the order of the identities (6)–(14).

3. Eighteen expansions

$$F^{(2)}[a;b,b';c,c';x,y]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r (b')_r}{r!\,(c)_r (c')_r} x^r y^r F(a+r,b+r;c+r;x)F(a+r,b'+r;c'+r;y), \tag{26}$$

$$F(a;b;c;x)F(a,b';c';y)$$

$$= \sum_{r=0}^{\infty} (-)^r \frac{(a)_r (b)_r (b')_r}{r!\,(c)_r (c')_r} x^r y^r F^{(2)}[a+r;b+r,b'+r;c+r,c'+r;x,y], \tag{27}$$

* We may note that with m, n positive integers the summations in the five lemmas become the finite sums from 0 to $\min(m,n)$.

$$F^{(3)}[a,a';b,b';c;x,y]$$
$$= \sum_{r=0}^{\infty} (-)^r \frac{(a)_r(a')_r(b)_r(b')_r}{r!\,(c+r-1)_r(c)_{2r}} x^r y^r F(a+r,b+r;c+2r;x) \times$$
$$\times F(a'+r,b'+r;c+2r;y), \quad (28)$$

$$F(a,b;c;x)F(a',b';c;y)$$
$$= \sum_{r=0}^{\infty} \frac{(a)_r(a')_r(b)_r(b')_r}{r!\,(c)_r(c)_{2r}} x^r y^r F^{(3)}[a+r,a'+r;b+r,b'+r;c+2r;x,y], \quad (29)$$

$$F^{(1)}[a;b,b';c;x,y]$$
$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(b')_r(c-a)_r}{r!\,(c+r-1)_r(c)_{2r}} x^r y^r F(a+r,b+r;c+2r;x) \times$$
$$\times F(a+r,b'+r;c+2r;y), \quad (30)$$

$$F(a,b;c;x)F(a,b';c;y)$$
$$= \sum_{r=0}^{\infty} (-)^r \frac{(a)_r(b)_r(b')_r(c-a)_r}{r!\,(c)_r(c)_{2r}} x^r y^r F^{(1)}[a+r;b+r,b'+r;c+2r;x,y], \quad (31)$$

$$F^{(1)}[a;b,b';c;x,y]$$
$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(b')_r}{r!\,(c)_{2r}} x^r y^r F^{(3)}[a+r,a+r;b+r,b'+r;c+2r;x,y], \quad (32)$$

$$F^{(3)}[a,a;b,b';c;x,y]$$
$$= \sum_{r=0}^{\infty} (-)^r \frac{(a)_r(b)_r(b')_r}{r!\,(c)_{2r}} x^r y^r F^{(1)}[a+r;b+r,b'+r;c+2r;x,y], \quad (33)$$

$$F^{(1)}[a;b,b';c;x,y]$$
$$= \sum_{r=0}^{\infty} (-)^r \frac{(a)_{2r}(b)_r(b')_r}{r!\,(c+r-1)_r(c)_{2r}} \times$$
$$\times x^r y^r F^{(2)}[a+2r;b+r,b'+r;c+2r,c+2r;x,y], \quad (34)$$

$$F^{(2)}[a;b,b';c,c;x,y]$$
$$= \sum_{r=0}^{\infty} \frac{(a)_{2r}(b)_r(b')_r}{r!\,(c)_r(c)_{2r}} x^r y^r F^{(1)}[a+2r;b+r,b'+r;c+2r;x,y], \quad (35)$$

$$F^{(4)}[a,b;c,c';x,y]$$
$$= \sum_{r=0}^{\infty} \frac{(a)_{2r}(b)_r}{r!\,(c)_r(c')_r} x^r y^r F^{(2)}[a+2r;b+r,b+r;c+r,c'+r;x,y]. \quad (36)$$

$$F^{(2)}[a;b,b;c,c';x,y]$$

$$= \sum_{r=0}^{\infty} (-)^r \frac{(a)_{2r}(b)_r}{r!\,(c)_r(c')_r} x^r y^r F^{(4)}[a+2r,b+r;c+r,c'+r;x,y], \quad (37)$$

$$F(a,b;c;x+y)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_{2r}(b)_r}{r!\,(c)_{2r}} x^r y^r F^{(1)}[a+2r;b+r,b+r;c+2r;x,y], \quad (38)$$

$$F^{(1)}[a;b,b;c;x,y]$$

$$= \sum_{r=0}^{\infty} (-)^r \frac{(a)_{2r}(b)_r}{r!\,(c)_{2r}} x^r y^r F(a+2r,b+r;c+2r;x+y), \quad (39)$$

$$F(a,b;c;x+y)$$

$$= \sum_{r=0}^{\infty} (-)^r \frac{(a)_{2r}(b)_{2r}}{r!\,(c+r-1)_r(c)_{2r}} \times$$
$$\times x^r y^r F^{(4)}[a+2r,b+2r;c+2r,c+2r;x,y], \quad (40)$$

$$F^{(4)}[a,b;c,c;x,y]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_{2r}(b)_{2r}}{r!\,(c)_r(c)_{2r}} x^r y^r F(a+2r,b+2r;c+2r;x+y), \quad (41)$$

$$F(a,b;c;x+y)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_{2r}(b)_r(c-b)_r}{r!\,(c+r-1)_r(c)_{2r}} x^r y^r F^{(2)}[a+2r;b+r,b+r;c+2r,c+2r;x,y], \quad (42)$$

$$F^{(2)}[a;b,b;c,c;x,y]$$

$$= \sum_{r=0}^{\infty} (-)^r \frac{(a)_{2r}(b)_r(c-b)_r}{r!\,(c)_r(c)_{2r}} x^r y^r F(a+2r,b+r;c+2r;x+y). \quad (43)$$

Since $F^{(4)}$, F are symmetrical in their parameters a, b, we can give variants of some of these formulae: for instance, (36), (37), (38).

Now we started with six distinct hypergeometric functions: $F(x+y)$, $F(x)F(y)$, and the four Appell's functions. If, then, we had been able to expand each function in a series of each of the other functions, we should have had thirty such series. As it is, we have so far given only eighteen. The other twelve would require the expansions of $\nabla(a)\nabla(b)$, $\nabla(a)\nabla(b)\Delta(c)$, $\nabla(a)\nabla(b)\nabla(c)$ and their inverses:

and these products of gamma functions do not appear to be simple-hypergeometric. We can, however, to some extent fill this gap with twelve other expansions in which the arguments of one of the functions are diminished by xy; these will include the expansion of $F^{(4)}$ with which our inquiry originated. We begin by stating these expansions (grouped in inverse pairs) and then proceed to their proof.

4. A further twelve expansions

$$F(a,b;c;x+y-xy)$$

$$= \sum_{r=0}^{\infty} (-)^r \frac{(a)_r (b)_r}{r!\,(c)_r} x^r y^r F(a+r,b+r;c+r;x+y), \tag{44}$$

$$F(a,b;c;x+y)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{r!\,(c)_r} x^r y^r F(a+r,b+r;c+r;x+y-xy), \tag{45}$$

$$F(a,b;c;x+y-xy)$$

$$= \sum_{r=0}^{\infty} (-)^r \frac{(a)_r (b)_r (c-a)_r}{r!\,(c)_{2r}} x^r y^r F^{(1)}[a+r;b+r,b+r;c+2r;x,y], \tag{46}$$

$$F^{(1)}[a;b,b;c;x,y]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r (c-a)_r}{r!\,(c)_{2r}} x^r y^r F(a+r,b+r;c+2r;x+y-xy), \tag{47}$$

$$F(a,b;c;x+y-xy)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r (a+b-c)_r}{r!\,(c)_{2r}} x^r y^r F^{(3)}[a+r,a+r;b+r,b+r;c+2r;x,y], \tag{48}$$

$$F^{(3)}[a,a;b,b;c;x,y]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r (c-a-b)_r}{r!\,(c)_{2r}} x^r y^r F(a+r,b+r;c+2r;x+y-xy), \tag{49}$$

$$F(a,b;c;x+y-xy)$$

$$= \sum_{r=0}^{\infty} (-)^r \frac{(a)_r (b)_r (c-a)_r (c-b)_r}{r!\,(c+r-1)_r (c)_{2r}} x^r y^r F(a+r,b+r;c+2r;x) \times$$
$$\times F(a+r,b+r;c+2r;y), \tag{50}$$

$$F(a,b;c;x)F(a,b;c;y)$$
$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(c-a)_r(c-b)_r}{r!\,(c)_r(c)_{2r}} x^r y^r F(a+r,b+r;c+2r;x+y-xy), \tag{51}$$

$$F^{(4)}[a,b;c,c';x-xy,y-xy]$$
$$= \sum_{r=0}^{\infty} (-)^r \frac{(a)_r(b)_r(c+c'-a-1)_r}{r!\,(c)_r(c')_r} \times$$
$$\times x^r y^r F^{(2)}[a+r;b+r,b+r;c+r,c'+r;x,y], \tag{52}$$

$$F^{(2)}[a;b,b;c,c';x,y]$$
$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(c+c'-a-1)_r}{r!\,(c)_r(c')_r} \times$$
$$\times x^r y^r F^{(4)}[a+r,b+r;c+r,c'+r;x-xy,y-xy], \tag{53}$$

$$F^{(4)}[a,b;c,c';x-xy,y-xy]$$
$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(a+b-c-c'+1)_r}{r!\,(c)_r(c')_r} \times$$
$$\times x^r y^r F(a+r,b+r;c+r;x)F(a+r,b+r;c'+r;y), \tag{54}$$

$$F(a,b;c;x)F(a,b;c';y)$$
$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(c+c'-a-b-1)_r}{r!\,(c)_r(c')_r} \times$$
$$\times x^r y^r F^{(4)}[a+r,b+r;c+r,c'+r;x-xy,y-xy], \tag{55}$$

To prove these we begin with (44), (45) which are disguised forms of Taylor's series. For we can write Taylor's series in the form

$$F(a,b;c;x-h) = \sum_{r=0}^{\infty} \frac{h^r}{r!} x^{-r}(-\delta)_r F(a,b;c;x)$$
$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{r!\,(c)_r}(-h)^r F(a+r,b+r;c+r;x), \quad \text{by (17).}$$

Replacing the arguments x, h by (i) $x+y$, xy and (ii) $x+y-xy$, $-xy$, we get (44), (45). In analogy with (19) we can rewrite (44) symbolically as

$$F(a,b;c;x+y-xy)$$
$$= {}_3F_2\left[\begin{matrix} 1-c-\delta-\delta', -\delta, -\delta' \\ 1-a-\delta-\delta', 1-b-\delta-\delta' \end{matrix}\right] F(a,b;c;x+y). \tag{56}$$

We can write the Taylor's series of $F^{(4)}(x-xy, y-xy)$ in similar symbolic form. For

$$F^{(4)}[a,b;c,c';x-xy,y-xy]$$

$$= \sum_{N=0}^{\infty} (xy)^N \sum_{r+s=N} \frac{x^{-r}y^{-s}}{r!\,s!}(-\delta)_r(-\delta')_s F^{(4)}[a,b;c,c';x,y]$$

$$= \sum_{N=0}^{\infty} (-xy)^N \sum_{r+s=N} \frac{(a)_N(b)_N}{r!\,s!\,(c)_r(c')_s} F^{(4)}[a+N,b+N;c+r,c'+s;x,y], \quad \text{by (17),}$$

$$= \sum_{N=0}^{\infty} \frac{(a)_N(b)_N}{(c)_N(c')_N}(-xy)^N \sum_{r+s=N} \frac{(c+N-s)_s(c'+N-r)_r}{s!\,r!} \times$$

$$\times F^{(4)}[a+N,b+N;c+N-s,c'+N-r;x,y].$$

But, on the analogy of (16),

$$\sum_{r+s=N} \frac{(c+N-s)_s(c'+N-r)_r}{s!\,r!} F^{(4)}[a+N,b+N;c+N-s,c'+N-r;x,y]$$

$$= \sum_{r+s=N} (-)^N \frac{(1-c-N-\delta)_s(1-c'-N-\delta')_r}{s!\,r!} \times$$

$$\times F^{(4)}[a+N,b+N;c+N,c'+N;x,y]$$

$$= (-)^N \frac{(2-c-c'-2N-\delta-\delta')_N}{N!} F^{(4)}[a+N,b+N;c+N,c'+N;x,y],$$

by Vandermonde's lemma. Thus

$$F^{(4)}[a,b;c,c';x-xy,y-xy]$$

$$= \sum_{N=0}^{\infty} \frac{(a)_N(b)_N}{(c)_N(c')_N} \frac{(2-c-c'-\delta-\delta')_N}{N!} \times$$

$$\times (xy)^N F^{(4)}[a+N,b+N;c+N,c'+N;x,y]$$

$$= \sum_{N=0}^{\infty} \frac{(2-c-c'-\delta-\delta')_N(-\delta)_N(-\delta')_N}{N!\,(1-a-\delta-\delta')_N(1-b-\delta-\delta')_N} F^{(4)}[a,b;c,c';x,y], \quad (57)$$

on the analogy of (19). Hence, finally,

$$F^{(4)}[a,b;c,c';x-xy,y-xy]$$

$$= {}_3F_2\left[\begin{matrix} 2-c-c'-\delta-\delta', & -\delta, & -\delta' \\ 1-a-\delta-\delta', & 1-b-\delta-\delta' \end{matrix}\right] F^{(4)}[a,b;c,c';x,y]. \quad (58)$$

This result does not, apparently, lead to an expansion like (44), but we use it as a basis from which to obtain the expansions involving $F^{(4)}$.

5. Two further lemmas

To complete the proof of the expansions of the preceding section we use two further lemmas, both due to Whipple,*

$$ {}_3F_2\left[\begin{matrix} h, & -m, & -n \\ a, & b & \end{matrix}\right] = \frac{\Gamma(a)\Gamma(a+m+n)}{\Gamma(a+m)\Gamma(a+n)}\, {}_3F_2\left[\begin{matrix} b-h, & -m, & -n \\ 1-a-m-n, & b & \end{matrix}\right], \quad (59) $$

$$ {}_6F_5\left[\begin{matrix} c-a, & c-b, & c-1, & \frac{1}{2}c+\frac{1}{2}, & -m, & -n; \\ a, & b, & \frac{1}{2}c-\frac{1}{2}, & c+m, & c+n & \end{matrix} -1\right] $$

$$ = \frac{\Gamma(c+m)\Gamma(c+n)}{\Gamma(c)\Gamma(c+m+n)}\, {}_3F_2\left[\begin{matrix} a+b-c, & -m, & -n \\ a, & b & \end{matrix}\right], \quad (60) $$

in which the hypergeometric series are terminating.

In applying these two lemmas we replace m, n by δ, δ' and rearrange so as to write them symbolically as

$$ {}_3F_2\left[\begin{matrix} b-h, & -\delta, & -\delta' \\ 1-a-\delta-\delta', & b & \end{matrix}\right] = {}_3F_2\left[\begin{matrix} h, & -\delta, & -\delta' \\ a, & b & \end{matrix}\right]\Delta(a), \quad (61) $$

$$ {}_3F_2\left[\begin{matrix} a+b-c, & -\delta, & -\delta' \\ a, & b & \end{matrix}\right] $$

$$ = {}_6F_5\left[\begin{matrix} c-a, & c-b, & c-1, & \frac{1}{2}c+\frac{1}{2}, & -\delta, & -\delta'; \\ a, & b, & \frac{1}{2}c-\frac{1}{2}, & c+\delta, & c+\delta' & \end{matrix} -1\right]\nabla(c). \quad (62) $$

Replacing a, b, h in (61) by b, $1-a-\delta-\delta'$, $c-a$ and applying it to (56) we get

$$ F(a,b;c;x+y-xy) = {}_3F_2\left[\begin{matrix} c-a, & -\delta, & -\delta' \\ 1-a-\delta-\delta', & b & \end{matrix}\right] F^{(1)}[a;b,b;c;x,y], \quad (63) $$

which is (46). Again, writing $h = a+b-c$ in (61) and applying it to (63) we get

$$ F(a,b;c;x+y-xy) = {}_3F_2\left[\begin{matrix} a+b-c, & -\delta, & -\delta' \\ a, & b & \end{matrix}\right] F^{(3)}[a,a;b,b;c;x,y], \quad (64) $$

which is (48). Applying (62) to (64) we get

$$ F(a,b;c;x+y-xy) $$

$$ = {}_6F_5\left[\begin{matrix} c-a, & c-b, & c-1, & \frac{1}{2}c+\frac{1}{2}, & -\delta, & -\delta'; \\ a, & b, & \frac{1}{2}c-\frac{1}{2}, & \delta+c, & \delta'+c & \end{matrix} -1\right] F(a,b;c;x)F(a,b;c;y), \quad (65) $$

which is (50).

* The first is covered by (7), 112 (23), quoted by Bailey (4), 22, § 3.9: it is the relation between $Fp(0)$ and $Fn(4)$, where $Fp(0)$ is $Fp(0;4,5)$ and $Fn(4)$ is $Fn(4;0,1)$. (For this reference we are indebted to the referee.) The second lemma is (8), 252 (6.3), quoted by Bailey (4), 28 (2).

Again, replacing a, b, h in (61) by b, $1-a-\delta-\delta'$, $c+c'-a-1$ and applying it to (58) we get

$$F^{(4)}[a,b;c,c';x-xy,y-xy]$$
$$= {}_3F_2\left[\begin{matrix}c+c'-a-1, & -\delta, & -\delta' \\ 1-a-\delta-\delta', & b & \end{matrix}\right] F^{(2)}[a;b,b;c,c';x,y], \quad (66)$$

which is (52). Finally writing $h = a+b-c-c'+1$ in (61) and applying it to (66) we get

$$F^{(4)}[a,b;c,c';x-xy,y-xy]$$
$$= {}_3F_2\left[\begin{matrix}a+b-c-c'+1, & -\delta, & -\delta' \\ & a, b & \end{matrix}\right] F(a,b;c;x)F(a,b;c';y),$$

which is (54). This completes the proof of the six 'direct' expansions. The other six expansions are obtained most quickly, perhaps, by simply inverting the direct expansions. Taking (55) as a sufficient example, we substitute for $F^{(4)}$ on the right from (54) and sum diagonally. This gives for the expression on the right

$$\sum_{N=0}^{\infty} \frac{(a)_N(b)_N}{(c)_N(c')_N}(xy)^N F(a+N,b+N;c+N;x)F(a+N,b+N;c'+N;y)\times$$
$$\times \sum_{r+s=N} \frac{(c+c'-a-b-1)_r(a+b-c-c'+1)_s}{r!\,s!}.$$

By Vandermonde's lemma the inner sum vanishes unless N vanishes, and then the sum is unity. Thus the expression on the right is just $F(a,b;c;x)F(a,b;c';y)$, and this proves (55). The other inverse expansions can be proved similarly.

So far all our proofs have been merely formal: for, as we have seen, the symbolic arguments can be interpreted to give formal term-by-term identification. We have still to justify the rearrangement of the double series by an appeal to their absolute convergence (or otherwise).

6. Integral expressions for $F^{(4)}$

Before, however, we proceed to this discussion of absolute convergence we remark that, if in the various expansions we replace F, $F^{(1)}$, $F^{(2)}$, $F^{(3)}$ by their known equivalents as single or double integrals we obtain integral equivalents (not necessarily new or interesting) for F and the $F^{(r)}$. We illustrate this by considering only integral equivalents for $F^{(4)}$.

We begin with (54) and on the right substitute for $F(x)$ from the formula

$$F(\alpha,\beta;\gamma;x)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}\,du \quad (67)$$

and for $F(y)$ from the analogous formula with α, β interchanged. Then we get that, if $\mathrm{R}(a)$, $\mathrm{R}(c-a)$, $\mathrm{R}(b)$, $\mathrm{R}(c'-b)$ are all positive,

$$F^{(4)}[a,b;c,c';x-xy,y-xy]$$

$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r(a+b-c-c'+1)_r}{r!\,(c)_r(c')_r}x^r y^r\,\frac{\Gamma(c+r)\Gamma(c'+r)}{\Gamma(a+r)\Gamma(c-a)\Gamma(b+r)\Gamma(c'-b)}\times$$

$$\times\int_0^1\int_0^1 u^{a+r-1}v^{b+r-1}(1-u)^{c-a-1}(1-v)^{c'-b-1}(1-ux)^{-(b+r)}(1-vy)^{-(a+r)}\,du\,dv$$

$$=\frac{\Gamma(c)\Gamma(c')}{\Gamma(a)\Gamma(c-a)\Gamma(b)\Gamma(c'-b)}\sum_{r=0}^{\infty}\int_0^1\int_0^1 u^{a-1}v^{b-1}(1-u)^{c-a-1}(1-v)^{c'-b-1}\times$$

$$\times(1-ux)^{-b}(1-vy)^{-a}\frac{(a+b-c-c'+1)_r}{r!}\left[\frac{uvxy}{(1-ux)(1-vy)}\right]^r du\,dv.$$

But, if $|x|\leqslant\rho$, $|y|\leqslant\rho'$, then

$$\left|\frac{uvxy}{(1-ux)(1-vy)}\right|\leqslant\frac{\rho\rho'}{(1-\rho)(1-\rho')}<1,\quad\text{if }\rho+\rho'<1.$$

Thus in this domain of x, y there is convergence, uniform in u, v over the region of integration. The order of summation and integration can therefore be reversed, and we have

$$F^{(4)}[a,b;c,c';x-xy,y-xy]=\frac{\Gamma(c)\Gamma(c')}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c'-b)}\times$$

$$\times\int_0^1\int_0^1 u^{a-1}v^{b-1}(1-u)^{c-a-1}(1-v)^{c'-b-1}(1-ux)^{a-c-c'+1}\times$$

$$\times(1-vy)^{b-c-c'+1}(1-ux-vy)^{c+c'-a-b-1}\,du\,dv, \quad (68)$$

valid when $\mathrm{R}(a)$, $\mathrm{R}(b)$, $\mathrm{R}(c-a)$, $\mathrm{R}(c'-b)>0$; $|x|\leqslant\rho$; $|y|\leqslant\rho'$; $\rho+\rho'<1$.*

* But, as pointed out by the referee, we should superimpose the condition $\sqrt{\{\rho(1+\rho')\}}+\sqrt{\{\rho'(1+\rho)\}}<1$, necessary for the convergence of the series defining $F^{(4)}$; and so for (69), (71).

If we had used (67) for the two substitutions in (54) without interchange of α, β, we should have obtained the alternative formula

$$F^{(4)}[a,b;c,c';x-xy,y-xy] = \frac{\Gamma(c)\Gamma(c')}{\{\Gamma(b)\}^2\Gamma(c-b)\Gamma(c'-b)}\times$$
$$\times\int_0^1\int_0^1 (uv)^{b-1}(1-u)^{c-b-1}(1-v)^{c'-b-1}(1-ux)^{-a}(1-vy)^{-a}\times$$
$$\times F\left(a,a+b-c-c'+1;b;\frac{uvxy}{(1-ux)(1-vy)}\right)dudv, \quad (69)$$

valid when $\mathrm{R}(b)$, $\mathrm{R}(c-b)$, $\mathrm{R}(c'-b) > 0$; $|x| \leqslant \rho$; $|y| \leqslant \rho'$; $\rho+\rho' < 1$. We can obtain variants of (69) by transformation of the hypergeometric function under the signs of integration.

If in (36) we substitute for $F^{(2)}$ from

$$F^{(2)}[\alpha;\beta,\beta';\gamma,\gamma';x,y] = \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')}\times$$
$$\times\int_0^1\int_0^1 u^{\beta-1}v^{\beta'-1}(1-u)^{\gamma-\beta-1}(1-v)^{\gamma'-\beta'-1}(1-ux-vy)^{-\alpha}\,dudv,$$

we get on reduction

$$F^{(4)}[a,b;c,c';x,y] = \frac{\Gamma(c)\Gamma(c')}{\{\Gamma(b)\}^2\Gamma(c-b)\Gamma(c'-b)}\times$$
$$\times\int_0^1\int_0^1 (uv)^{b-1}(1-u)^{c-b-1}(1-v)^{c'-b-1}(1-ux-vy)^{-a}\times$$
$$\times F\left(\tfrac{1}{2}a,\tfrac{1}{2}a+\tfrac{1}{2};b;\frac{4uvxy}{(1-ux-vy)^2}\right)dudv, \quad (70)$$

valid when

$$\mathrm{R}(b),\ \mathrm{R}(c-b),\ \mathrm{R}(c'-b) > 0;\ |x| \leqslant \rho;\ |y| \leqslant \rho';\ \sqrt{\rho}+\sqrt{\rho'} < 1.$$

If we make a similar substitution for $F^{(2)}$ in (52), we obtain the formula

$$F^{(4)}[a,b;c,c';x-xy,y-xy] = \frac{\Gamma(c)\Gamma(c')}{\{\Gamma(b)\}^2\Gamma(c-b)\Gamma(c'-b)}\times$$
$$\times\int_0^1\int_0^1 (uv)^{b-1}(1-u)^{c-b-1}(1-v)^{c'-b-1}(1-ux-vy)^{-a}\times$$
$$\times F\left(a,c+c'-a-1;b;-\frac{uvxy}{1-ux-vy}\right)dudv, \quad (71)$$

valid when

$$\mathrm{R}(b),\ \mathrm{R}(c-b),\ \mathrm{R}(c'-b) > 0;\ |x| \leqslant \rho;\ |y| \leqslant \rho';\ \rho+\rho'+\rho\rho' < 1.$$

Finally, if in (41) we substitute for F from (67), interchanging the parameters α, β, we get on reduction

$$F^{(4)}[a,b;c,c;x,y] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \times$$

$$\times \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx-ty)^{-a} F\left[\tfrac{1}{2}a, \tfrac{1}{2}a+\tfrac{1}{2}; c; \frac{4xyt^2}{(1-tx-ty)^2}\right] dt, \quad (72)$$

valid when

$$\mathrm{R}(b),\ \mathrm{R}(c-b) > 0;\ |x| < \rho;\ |y| < \rho';\ \sqrt{\rho}+\sqrt{\rho'} < 1.$$

It will be noticed that here the function $F^{(4)}$ has two parameters equal.

7. Bounds of the hypergeometric functions

We conclude by establishing the absolute convergence of the various expansions we have already proved formally. In this formal proof we have, in effect, expanded the simple series on the right into triple series, and it is these triple series that we have to show absolutely convergent. Replacing each term in them by its absolute value, we incidentally replace the double hypergeometric series by their series of absolute values. Now, if a, α, c, γ are real, we have inequalities of the type

$$|a+i\alpha+m| < |a|+|\alpha|+m,$$

and

$$\frac{1}{|c+i\gamma+m|} < \frac{1}{c+m-|\gamma|} \quad \text{if } c > |\gamma|.$$

But, since the parameters occur in the series with increments r, $2r$, we can sufficiently suppose that their real parts are as large as we please, and, in particular, that $c > |\gamma|$.

We may thus take all the parameters in the hypergeometric functions to be real and positive; the variables x, y should be replaced by their absolute values $|x|$, $|y|$, but, for simplicity, we shall for the moment merely regard x, y themselves as positive, replacing them by $|x|$, $|y|$ in the final statement of the convergence conditions, which involve x, y alone.

What we have said above applies strictly only to the expansions of § 3. In those of § 4, where functions of argument $x-xy$, $y-xy$, $x+y-xy$ appear, the ultimate series will be quadruple, proceeding in powers of x, y, xy, xy. In converting these quadruple series into

their series of absolute values, we must take the further precaution of replacing the three arguments just mentioned by $|x|+|xy|$, $|y|+|xy|$, $|x|+|y|+|xy|$, and we shall therefore need, for these expansions, conditions that will secure the convergence of the hypergeometric series of these arguments as well as that of the single series on the right.

We need then to know simple bounds for the hypergeometric functions with positive variables when their (positive) parameters diverge to infinity in certain ways. We establish these bounds by first proving the following simple inequalities:

(i) $F(a+1,b;c+1;x) \leqslant \dfrac{c}{A} F(a,b;c;x)$,

where $A \equiv \min(a,c)$;

(ii) $(1-x)F(a+1,b+1;c+1;x) \leqslant \dfrac{c}{A} F(a,b;c;x)$,

where $A \equiv \min(a,b,c)$;

(iii) $(1-x)F^{(1)}[a+1;b+1;b';c+1;x,y] < \dfrac{c}{b} F^{(1)}[a;b,b';c;x,y]$

if $a,c > b$; (73)

(iv) $(1-xy)F^{(1)}[a+1;b+1,b'+1;c+2;x,y]$

$$< \frac{c(c+1)}{aA} F^{(1)}[a;b,b';c;x,y]$$

where $A \equiv \min(a,b,b')$, if $c > a$; (74)

(v) $(1-x)F^{(2)}[a+1;b+1,b';c+1,c'+1;x,y]$

$$< \frac{cc'}{ab} F^{(2)}[a;b,b';c;x,y],$$

if $cc' > ab$, $c+c' > a+b$, $c > b$; (75)

(vi) $(1-x-y)F^{(2)}[a+1;b+1,b'+1;c+1,c'+1;x,y]$

$$\leqslant \frac{cc'}{BB'} F^{(2)}[a;b,b';c;x,y],$$

where $B \equiv \min(a,b,c)$, $B' \equiv \min(a,b',c')$;

(vii) $(1-x-y)F^{(2)}[a+2;b+1,b'+1;c+1,c'+1;x,y]$

$$< \frac{A+1}{a+1} F^{(2)}[a+1;b+1,b'+1;c+1,c'+1;x,y],$$

where $A \equiv \max(a, a+b-c, a+b'-c', a+b+b'-c-c')$;

(viii) $F^{(3)}[a+1,b;a',b'+1;c+1;x,y] < \dfrac{c}{B} F^{(3)}[a,b;a',b';c;x,y]$,

where $B \equiv \min(b', c-b)$;

(ix) $(1-x-y)F^{(4)}[a+1,b+1;c+1,c'+1;x,y]$

$$< \frac{2cc'}{ab} F^{(4)}[a,b;c,c';x,y],$$

if $2cc' > ab$ and $2c+c', c+2c' > a+b$. (76)

8. Proof of the inequalities

In each case we compare the ratios of corresponding coefficients on the two sides. More precisely, we denote by R_m (or $R_{m,n}$) the ratio of the coefficient of x^m (or $x^m y^n$) on the left to the corresponding coefficient on the right and show that $R_m < 1$ (or $R_{m,n} < 1$). It may be noted that the factor $1-x-y$ where it occurs on the left is positive in virtue of the conditions necessary for the convergence of the series for $F^{(2)}$, $F^{(4)}$.

(i) $R_m = \dfrac{A(a+m)}{a(c+m)} \leqslant 1$, since $A \leqslant a, c$;

(ii) $R_m = \dfrac{A}{ab}\left\{\dfrac{(a+m)(b+m)}{c+m} - m\right\} = \dfrac{Aab+A(a+b-c)m}{abc+abm} \leqslant 1,$

since $c \geqslant A$ and

$$ab-A(a+b-c) = (a-A)(b-A)+A(c-A) \geqslant 0;$$

(iii) $R_{m,n} = \dfrac{(a+m+n)(b+m)}{a(c+m+n)} - \dfrac{m}{a}$

$$= \frac{ab+(a+b-c)m+bn}{ac+am+an} < 1, \text{ by (73)};$$

(iv) $R_{m,n} = \dfrac{A}{bb'}\left\{\dfrac{(a+m+n)(b+m)(b'+n)}{(c+m+n)(c+m+n+1)} - \dfrac{mn}{a+m+n-1}\right\}$

$$< \frac{A\{(b+m)(b'+n)-mn\}}{bb'(a+m+n)}, \text{ since } c > a,$$

$$= \frac{A(bb'+b'm+bn)}{bb'(a+m+n)} < 1, \text{ since } A < a, b, b';$$

(v) $R_{m,n} = \dfrac{(a+m+n)(b+m)}{(c+m)(c'+n)} - \dfrac{m}{c'+n}$

$$= \frac{ab+(a+b-c)m+bn+mn}{cc'+c'm+cn+mn} < 1, \text{ by (75)};$$

(vi) $$R_{m,n} = \frac{BB'}{abb'}\left\{\frac{(a+m+n)(b+m)(b'+n)}{(c+m)(c'+n)} - \frac{m(b'+n)}{c'+n} - \frac{n(b+m)}{c+m}\right\}$$

$$= \frac{BB'\{abb'+b'(a+b-c)m+b(a+b'-c')n+(a+b+b'-c-c')mn\}}{abb'(c+m)(c'+n)}$$

$$\leqslant 1,$$

since $B,\ B' \leqslant c,\ c';\ B(a+b-c) \leqslant ab,\ B'(a+b'-c') \leqslant ab'$ as in (ii); and so $BB'(a+b+b'-c-c') \leqslant a(bB'+b'B-BB') \leqslant abb'$, since $(b-B)(b'-B') \geqslant 0$;

(vii) $$R_{m,n} = \frac{1}{A+1}\left\{a+m+n+1-\frac{m(c+m)}{b+m} - \frac{n(c'+n)}{b'+n}\right\}$$

$$= \frac{1}{A+1}\left\{a+1+\frac{m(b-c)}{b+m} + \frac{n(b'-c')}{b'+n}\right\} < 1,$$

since $$\frac{m(b-c)}{b+m} + \frac{n(b'-c')}{b'+n'} < \max(0, b-c, b'-c', b-c+b'-c');$$

(viii) We prove the stronger inequality

$$BF^{(3)}[a+1, b; a', b'+1; c+1; x, y] +$$

$$+b(1-x)F^{(3)}[a+1, b+1; a', b'; c+1; x, y]$$

$$< c\,F^{(3)}[a, b; a', b'; c; x, y]:$$

here

$$R_{m,n} = \frac{(B/b')(a+m)(b'+n)+(a+m)(b+m)-m(c+m+n)}{a(c+m+n)}$$

$$\leqslant 1+\frac{(a+m)(B+n+b+m-c-m-n)}{a(c+m+n)}$$

$$\leqslant 1, \quad \text{since } B+b \leqslant c;$$

(ix) $$R_{m,n} = \frac{(a+m+n)(b+m+n)-m(c+m)-n(c'+n)}{2(c+m)(c'+n)}$$

$$= \frac{ab+(a+b-c)m+(a+b-c')n+2mn}{2(cc'+c'm+cn+mn)}$$

$$< 1, \quad \text{by (76).}$$

9. The convergence conditions

By repeated application of these inequalities (if necessary, with interchange of parameters) we get*

$$F(a+r,b+r;c+2r;x) < \frac{(c)_{2r}}{(a)_r(b)_r} F(a,b;c;x), \tag{77}$$

$$F(a+r,b+r;c+r;x) \leqslant \frac{(c)_r}{(A)_r}(1-x)^{-r}F(a,b;c;x) \tag{78}$$

where $A \equiv \min(a,b,c)$, and, by combining (i), (ii),

$$F(a+2r,b+r;c+2r;x) < \frac{(c)_{2r}}{(A)_{2r}}(1-x)^{-r}F(a,b;c;x), \tag{79}$$

where $A \equiv \min(a,c)$. Continuing we have

$$F^{(1)}[a+2r;b+r,b'+r;c+2r;x,y]$$

$$< \frac{(c)_{2r}}{(b)_r(b')_r}(1-x)^{-r}(1-y)^{-r}F^{(1)}[a;b,b';c;x,y], \tag{80}$$

$$F^{(1)}[a+r;b+r,b'+r;c+2r;x,y]$$

$$< \frac{(c)_{2r}}{(a)_r(A)_r}(1-xy)^{-r}F^{(1)}[a;b,b';c;x,y], \tag{81}$$

where $A \equiv \min(a,b,b')$,

$$F^{(2)}[a+2r;b+r,b'+r;c+2r,c'+2r;x,y]$$

$$< \frac{(c)_{2r}(c')_{2r}}{(a)_{2r}(b)_r(b')_r}(1-x)^{-r}(1-y)^{-r}F^{(2)}[a;b,b';c,c';x,y], \tag{82}$$

$$F^{(2)}[a+r;b+r,b'+r;c+r,c'+r;x,y]$$

$$\leqslant \frac{(c)_r(c')_r}{(B)_r(B')_r}(1-x-y)^{-r}F^{(2)}[a;b,b';c,c';x,y], \tag{83}$$

where $B \equiv \min(a,b,c)$, $B' \equiv \min(a,b',c')$,
and, by combining (vi), (vii),

$$F^{(2)}[a+2r;b+r,b'+r;c+r,c'+r;x,y]$$

$$< \frac{(A)_{2r}(c)_r(c')_r}{(a)_{2r}(B)_r(B')_r}(1-x-y)^{-2r}F^{(2)}[a;b,b';c,c';x,y], \tag{84}$$

* Since $c+2r$ diverges to infinity more rapidly than $a+r$, $b+r$, we may sufficiently suppose in (77) that $c > a, b$, and we make analogous assumptions in what follows.

where $B \equiv \min(b, c)$, $B' \equiv \min(b', c')$, $A \equiv \max(a+b-B, a+b'-B')$. Proceeding, we get

$$F^{(3)}[a+r, b+r; a'+r, b'+r; c+2r; x, y] < \frac{(c)_{2r}}{(B)_r (B')_r} F^{(3)}[a, b; a', b'; c; x, y], \tag{85}$$

where $B \equiv \min(b', c-b)$, $B' \equiv \min(b, c-b')$,

$$F^{(4)}[a+r, b+r; c+r, c'+r; x, y] < \frac{(c)_r (c')_r}{(a)_r (b)_r} 2^r (1-x-y)^{-r} F^{(4)}[a, b; c, c'; x, y]. \tag{86}$$

To these we add an inequality for $F^{(4)}[a+2r, b+r; c+r, c'+r; x, y]$, which we obtain, less simply, as follows. From (36) we have

$$F^{(4)}[a+2R, b+R; c+R, c'+R; x, y] = \sum_{r=0}^{\infty} \frac{(a+2R)_{2r} (b+R)_r}{r!\,(c+R)_r (c'+R)_r} \times$$

$$\times x^r y^r F^{(2)}[a+2R+2r; b+R+r, b+R+r; c+R+r, c'+R+r; x, y]$$

$$< \sum_{r=0}^{\infty} \frac{(a+2R)_{2r} (b+R)_r}{r!\,(c+R)_r (c'+R)_r} x^r y^r \frac{(A)_{2R+2r} (c)_{R+r} (c')_{R+r}}{(a)_{2R+2r} (B)_{R+r} (B')_{R+r}} (1-x-y)^{-(2R+2r)} \times$$

$$\times F^{(2)}[a; b, b; c, c'; x, y], \quad \text{by (84)},$$

$$= \frac{(A)_{2R} (c)_R (c')_R}{(a)_{2R} (B)_R (B')_R} (1-x-y)^{-2R} F^{(2)}[a; b, b; c, c'; x, y] \times$$

$$\times \sum_{r=0}^{\infty} \frac{(A+2R)_{2r} (b+R)_r}{r!\,(B+R)_r (B'+R)_r} \left\{\frac{xy}{(1-x-y)^2}\right\}^r. \tag{87}$$

Now
$$\frac{(\frac{1}{2}A+m)(\frac{1}{2}A+\frac{1}{2}+m)(b+m)}{(B+m)(B'+m)} \leqslant H+m,$$

where

$$H \equiv \max\left\{A+b+\tfrac{1}{2}-B-B', \frac{A^2+A+4AB+2b-4BB'}{4(B+B')}, \frac{bA(A+1)}{4BB'}\right\},$$

and so the infinite sum in (87) does not exceed

$$\sum_{r=0}^{\infty} \frac{(H+R)_r}{r!} \left\{\frac{4xy}{(1-x-y)^2}\right\}^r.$$

Thus

$$F^{(4)}[a+2R; b+R; c+R, c'+R; x, y] < \frac{(A)_{2R} (c)_R (c')_R}{(a)_{2R} (B)_R (B')_R} (1-2x-2y+x^2-2xy+y^2)^{-(H+R)} (1-x-y)^{2H} \times$$

$$\times F^{(2)}[a; b, b; c, c'; x, y]. \tag{88}$$

Using these asymptotic forms in the thirty expansions, we get the following regions of validity:

$$|x|,\ |y| < 1 \text{ in (28), (29), (30), (32);}$$

$$|x|+|y| < 1 \text{ in (26), (34), (35), (38), (42);}$$

$$|\sqrt{x}|+|\sqrt{y}| < 1 \text{ in (36), (40), (41);}$$

$$|xy| < \tfrac{1}{2} \text{ in (31), (33);}$$

$$|x|+|y|+|xy| < 1 \text{ in (27), (39), (43), (44), (46)–(51);}$$

$$|x|+2|\sqrt{(2xy)}|+|y| < 1 \text{ in (37);}$$

$$|x|+|y|+2|xy| < 1 \text{ in (45);}$$

$$\sqrt{\{|x|(1+|y|)\}}+\sqrt{\{|y|(1+|x|)\}} < 1 \text{ in (52)–(55).}$$

We recall that $|x|,\ |y| < 1$; $|x|+|y| < 1$; $|\sqrt{x}|+|\sqrt{y}| < 1$ are the regions of convergence of the series defining $F^{(1)}$, $F^{(3)}$; $F^{(2)}$; $F^{(4)}$ respectively.

The asymptotic forms of this section can also be obtained if we give the hypergeometric functions their representation as single or double integrals, but then only under restrictive conditions of the type $\mathrm{R}(c-b) > 0$, $\mathrm{R}(c-b-b') > 0$. This method confirms the inequalities given above except in the case of (81) and (85) which it replaces by

$$F^{(1)}[a+r;b+r,b'+r;c+2r;x,y]$$

$$< \frac{(c)_{2r}}{(b)_r(b')_r}\{\sqrt{(1-x)}+\sqrt{(1-y)}\}^{-2r}F^{(1)}[a;b,b';c;x,y] \qquad (89)$$

and

$$F^{(3)}[a+r,b+r;a'+r,b'+r;c+2r;x,y]$$

$$< \frac{(c)_{2r}}{(b)_r(b')_r}\{1+\sqrt{[(1-x)(1-y)]}\}^{-2r}F^{(3)}[a,b;a',b';c;x,y], \quad (90)$$

both under the condition $\mathrm{R}(c-b-b') > 0$. We have not thought it worth while to study this point more closely.

10. Generalization

We have not succeeded in extending these results to analogous second-order hypergeometric functions in more than two variables, but it is not difficult to apply the arguments to functions of higher

order (i.e. with more parameters) in two variables. For instance, define

$$
\begin{aligned}
{}_{p+1}F_p^{(2)}\left[\begin{matrix} a: & b_1,\dots, & b_p; & b'_1,\dots, & b'_p; \\ & c_1,\dots, & c_p; & c'_1,\dots, & c'_p; \end{matrix} x, y\right] & \\
&\equiv \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m\dots(b_p)_m(b'_1)_n\dots(b'_p)_n}{m!\,n!\,(c_1)_m\dots(c_p)_m(c'_1)_n\dots(c'_p)_n} x^m y^n \\
&= \nabla(a)\,{}_{p+1}F_p\left[\begin{matrix} a, & b_1,\dots, & b_p; \\ & c_1,\dots, & c_p \end{matrix} x\right] {}_{p+1}F_p\left[\begin{matrix} a, & b'_1,\dots, & b'_p; \\ & c'_1,\dots, & c'_p \end{matrix} y\right]. \qquad (91)
\end{aligned}
$$

In analogy with (26) we shall have

$$
\begin{aligned}
{}_{p+1}F_p^{(2)}\left[\begin{matrix} a: & b_1,\dots, & b_p; & b'_1,\dots, & b'_p; \\ & c_1,\dots, & c_p; & c'_1,\dots, & c'_p; \end{matrix} x, y\right] & \\
= \sum_{r=0}^{\infty} \frac{(a)_r(b_1)_r\dots(b_p)_r(b'_1)_r\dots(b'_p)_r}{r!\,(c_1)_r\dots(c_p)_r(c'_1)_r\dots(c'_p)_r} x^r y^r\, {}_{p+1}F_p\left[\begin{matrix} a+r, & b_1+r,\dots, & b_p+r; \\ & c_1+r,\dots, & c_p+r \end{matrix} x\right] \times & \\
\times\, {}_{p+1}F_p\left[\begin{matrix} a+r, & b'_1+r,\dots, & b'_p+r; \\ & c'_1+r,\dots, & c'_p+r \end{matrix} y\right] \qquad (92) &
\end{aligned}
$$

in a suitable region of convergence.

REFERENCES

1. P. Appell and J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques* (Gauthier-Villars, 1926).
2. W. N. Bailey, *Quart. J. of Math.* (Oxford), 4 (1933), 305–8.
3. —— ibid., 5 (1934), 291–2.
4. —— *Generalized Hypergeometric Series* (Cambridge, 1935).
5. J. Dougall, *Proc. Edinburgh Math. Soc.* 25 (1907), 114–32.
6. L. Saalschütz, *Zeits. für Math. u. Phys.* 36 (1891), 278–95.
7. F. J. W. Whipple, *Proc. London Math. Soc.* (2), 23 (1925), 104–14.
8. —— ibid., 24 (1926), 247–63.

A BOUNDARY PROBLEM

By B. SPAIN (*Edinburgh*)

[Received 27 July 1940]

IN this note the following theorem is proved:

Given a closed curve C, consisting of a finite number of smooth curves, which is the boundary of a closed domain D, and a function $f(x,y)$ such that* (i) $f = 0$ *on* C, (ii) $\operatorname{grad} f = 0$ *on* C, (iii) $\Delta f \geqslant 0$ *on* C,† *the equality sign only holding at most at a finite number of isolated points,* (iv) $\Delta\Delta f \geqslant 0$ *in* D, (v) f *and all its derivatives up to and including the third order are continuous at any point of* D, *then* $f \geqslant 0$ *in* D, *the equality sign not holding in any sub-domain where* $\Delta\Delta f > 0$.

LEMMA 1.‡ *If* $g(x,y) = 0$ *on* C, $\Delta g \leqslant 0$ *in* D, *and* g *and its first derivatives are continuous in* D, *then* $g \geqslant 0$ *in* D, *the equality sign not holding in any sub-domain where* $\Delta g < 0$.

LEMMA 2.§ *In any domain where* $\Delta g(x,y)$ *is negative* (*positive*), g *cannot have a minimum* (*maximum*).

LEMMA 3. *If* $\partial/\partial n\{g(x,y)\} = 0$ *on* C, *and* g *and its first derivatives are continuous in* D, *then* Δg *must take both positive and negative values in* D, *or else* $\Delta g \equiv 0$ *in* D.

The proof follows immediately from Gauss's theorem.

LEMMA 4. *If* $g(x,y)$ *and its derivatives up to and including the second order be continuous, and* $\operatorname{grad} g = 0$ *on* C, *then* g_{xx}, g_{yy}, *and* Δg *have the same sign at any point of* C.

Differentiating the relations $g_x = 0$, $g_y = 0$, we obtain

$$g_{xx} + g_{xy}y' = 0 \quad \text{and} \quad g_{xy} + g_{yy}y' = 0,$$

where the dash denotes differentiation with respect to x. Because the curves are smooth, y' is single-valued and thus $g_{xx}g_{yy} = g_{xy}^2$. The lemma then follows immediately.

Since $\partial f/\partial n = 0$ on C, Δf must take both positive and negative values in D by Lemma 3. (The trivial and obvious case $\Delta f \equiv 0$ will

* A smooth curve is here a curve possessing a continuous tangent.

† Δ is the two-dimensional Laplacian operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$.

‡ Cf. Frank-Mises, *Differentialgleichungen der Physik*, 1 (1930), 698.

§ Cf. W. D. MacMillan, *Theoretical Mechanics—Theory of the Potential* (New York, 1930), 135, Theorem IV.

be neglected in the discussion.) Thus there exist curves in D on which $\Delta f = 0$. From (iii) these curves must be closed, lying wholly in D. From (iv) and Lemma 1, $\Delta f \leqslant 0$ inside these closed curves. Thus there exists an interior domain D_i, bounded by closed curves C_i on which $\Delta f = 0$, in which $\Delta f \leqslant 0$ and an exterior domain D_e, bounded by C_i and C, in which $\Delta f \geqslant 0$.

Since $\operatorname{grad} f = 0$ on C, f attains an extreme value on C, being a maximum by Lemma 4 and (iii). Using (i), we see that there exists a sub-domain D' of D bounded by C and some interior closed curves in which $f \geqslant 0$. By Lemma 2 there cannot exist a maximum in D_e, from which we conclude that D' at least includes all D_e, i.e. $f \geqslant 0$ in D_e. If $f \leqslant 0$ in any portion of D_i, we must have a closed sub-domain of D_i in which $f \leqslant 0$, $\Delta f \leqslant 0$, and on whose boundary $f = 0$ in contradiction to Lemma 1. Thus we conclude that $f \geqslant 0$ in D as required.

The generalization to any number of dimensions is straightforward and follows immediately.

This theorem has the following interesting statical application: 'An elastic plate with clamped horizontal edge, such that the bending moment is positive* at the boundary, acted on by any system of forces vertically downwards can have no part of the plate above the plane of the edge after equilibrium has set in. Also Green's function for such a plate does not change sign.' From the point of view of applications, the assumption (iii) is not satisfactory as it is not always easy to see if this condition is satisfied. It should be possible to replace (iii) by some other assumption. This theorem restricted to convex domains but without (iii) has been communicated by Dr. J. Barta to Dr. A. Erdélyi without proof. Probably the condition that D be convex is sufficient, though certainly not necessary. Condition (iii) is a more general assumption, but it is well to note that it is also only a sufficient condition. A necessary condition is $\Delta f \geqslant 0$ on C without restricting the equality sign to isolated points.

* Cf. S. P. Timoshenko, *Theory of Elastic Stability* (New York and London, 1936), 287.

AN INEQUALITY IN THE TRIANGLE

By G. N. WATSON (*Birmingham*)

[Received 1 September 1940]

THIS note is concerned with the following property of a triangle:

Let ABC be any triangle and let P be any point inside it; let AP, BP, CP produced meet the opposite sides of the triangle at A', B', C' respectively. Then at least one of the inequalities

$$\frac{AA'}{BC} \geqslant \frac{\sqrt{3}}{2}, \qquad \frac{BB'}{CA} \geqslant \frac{\sqrt{3}}{2}, \qquad \frac{CC'}{AB} \geqslant \frac{\sqrt{3}}{2}$$

holds.

As was pointed out to me by Mr. T. W. Chaundy, this theorem is included in the following theorem:

That one of the above inequalities which involves the shortest of the three sides of the triangle is true.

It will be seen presently that this theorem can be proved very simply either geometrically or algebraically. The theorem first stated is, however, also included in the following theorem:

That one of the above inequalities which involves the longest of the three transversals AA', BB', CC' is true.

This theorem is distinctly deeper than either of the others. It was brought to my notice some years ago, when I was told that no proof of it was known and that it was of some importance in connexion with an investigation on the Problem of Three Bodies.

In this note I give reasonably simple proofs of both the trivial and the deep theorems. It is to be remarked at the outset that, as soon as either theorem has been proved, it becomes evident that the constant $\frac{1}{2}\sqrt{3}$ is 'best possible', because it is attained by taking the triangle to be equilateral with P at the circumcentre.

Let the length of the sides of the triangle ABC be a, b, c, and let the actual areal coordinates of P be (x, y, z), so that x, y, z are all positive.

Evidently A' divides BC in the ratio $z:y$, and hence, from the elementary geometry of the triangle,

$$AA'^2 = \frac{zb^2+yc^2}{y+z} - \frac{yza^2}{(y+z)^2}. \tag{1}$$

In like manner we have

$$BB'^2 = \frac{xc^2+za^2}{z+x} - \frac{zxb^2}{(z+x)^2}, \tag{2}$$

$$CC'^2 = \frac{ya^2+xb^2}{x+y} - \frac{xyc^2}{(x+y)^2}. \tag{3}$$

We now discuss the deeper theorem. There is clearly no loss of generality in supposing AA' to be the longest transversal, so that we have

$$AA'^2 \geqslant BB'^2, \tag{4}$$

$$AA'^2 \geqslant CC'^2, \tag{5}$$

and we now have to prove that $AA'/a \geqslant \frac{1}{2}\sqrt{3}$. The substance of my proof is the deduction from the equations and inequalities (1)–(5) of an inequality which contains AA' and a, regardless of the shape of the triangle ABC, heed being paid only to the areal coordinates of P. In other words, we seek to eliminate b, c, BB', CC' from the five equations and inequalities, and then see what can be made of the resulting inequality. Since the five questions and inequalities are all linear in the variables b^2, c^2, BB'^2, CC'^2, it is easy enough to effect the elimination, provided only that (as turns out to be the case) the directions of the inequalities (4) and (5) are such that the inequalities can be combined in the desired manner.

When we eliminate BB' and CC' from (2)–(5) and replace AA' by its value given in (1), we get

$$\left(\frac{z}{y+z}+\frac{zx}{(z+x)^2}\right)b^2+\left(\frac{y}{y+z}-\frac{x}{z+x}\right)c^2 \geqslant \left(\frac{z}{z+x}+\frac{yz}{(y+z)^2}\right)a^2,$$

$$\left(\frac{z}{y+z}-\frac{x}{x+y}\right)b^2+\left(\frac{y}{y+z}+\frac{xy}{(x+y)^2}\right)c^2 \geqslant \left(\frac{y}{x+y}+\frac{yz}{(y+z)^2}\right)a^2.$$

After slight simplifications these inequalities reduce to

$$(y+z)(x^2+z^2+3zx+xy)b^2+(y-x)(y+z)(z+x)c^2 \geqslant (z+x)(y^2+z^2+3yz+xy)a^2, \tag{6}$$

$$(z-x)(y+z)(x+y)b^2+(y+z)(x^2+y^2+zx+3xy)c^2 \geqslant (x+y)(y^2+z^2+3yz+zx)a^2. \tag{7}$$

We now have to eliminate b and c from (1), (6), and (7). To do this, we endeavour to combine (6) and (7) linearly in such a way that b and c occur in the combination only in the form zb^2+yc^2. It turns out that we attain this end by adding (6) and (7) just as they stand; we thus get

$$(y+z)(4x+y+z)(zb^2+yc^2)$$
$$\geqslant \{x^2(y+z)+2x(y^2+z^2+4yz)+(y+z)(y^2+z^2+3yz)\}a^2,$$

and hence, by introducing this inequality into (1),

$$(y+z)^2(4x+y+z)AA'^2$$
$$\geqslant \{x^2(y+z)+2x(y^2+z^2+4yz)+(y+z)(y^2+z^2+3yz)\}a^2-$$
$$-(4x+y+z)yza^2$$
$$= (y+z)(x+y+z)^2a^2.$$

We thus have

$$\frac{AA'^2}{a^2} \geqslant \frac{(x+y+z)^2}{(y+z)(4x+y+z)}$$
$$= \frac{3}{4}+\frac{(2x-y-z)^2}{4(y+z)(4x+y+z)} \geqslant \frac{3}{4},$$

whence the truth of the deeper theorem is obvious.

We next give an algebraical proof of the simpler theorem. Assuming, as we may without loss of generality, that a is now the shortest side, so that $b \geqslant a$ and $c \geqslant a$, we have from (1)

$$AA'^2 \geqslant \frac{za^2+ya^2}{y+z}-\frac{yza^2}{(y+z)^2}$$
$$= \left(\frac{3}{4}+\frac{(y-z)^2}{4(y+z)^2}\right)a^2 \geqslant \frac{3a^2}{4},$$

whence the truth of the simpler theorem is obvious.

Finally, the following geometrical proof of the simpler theorem was pointed out to me by Mr. Chaundy, who remarked that in this theorem 'the concurrence at P is to be regarded as a red herring'.

As before, we take a to be the shortest side, and we now further take c to be the longest side, so that $a \leqslant b \leqslant c$ and therefore $A \leqslant B \leqslant C$. Then $3C \geqslant A+B+C = 180°$, so that $C \geqslant 60°$. We now consider two different ranges of values of C.

(i) First let $60° \leqslant C \leqslant 120°$, and let p be the length of the perpendicular from A to BC. Then

$$\frac{AA'}{a} \geqslant \frac{p}{a} = \frac{b \sin C}{a} \geqslant \sin C \geqslant \sin 60° = \frac{\sqrt{3}}{2}.$$

(ii) Next let $90° \leqslant C \leqslant 180°$. Then

$$\frac{AA'}{a} \geqslant \frac{b}{a} \geqslant 1 > \frac{\sqrt{3}}{2}.$$

Hence the required inequality holds for all values of C in the overlapping ranges $60° \leqslant C \leqslant 120°$, $90° \leqslant C \leqslant 180°$, that is, for all possible values of C. This establishes the simpler theorem.

ON LEVEL CURVES OF INTEGRAL FUNCTIONS

By M. L. CARTWRIGHT (*Cambridge*)

[Received 27 September 1940]

1. LET $f(z)$ be an integral or meromorphic function; a curve on which $|f(z)|$ is constant is called a *level curve* of $f(z)$. Level curves, or branches of level curves, which extend continuously to infinity are sometimes called *open* level curves and the others *closed* level curves. Any closed level curve contains at least one simple closed curve as a sub-set; and a simple closed curve on which $|f(z)| = M$ encloses a simply-connected domain D in which $|f(z)| < M$. Further, $f(z)$ takes every value α for which $|\alpha| < M$ at least once in D.

The chief object of this paper is to obtain some results for open level curves of integral functions corresponding to those about integral functions which have a common closed level curve. The problem for closed level curves was suggested by J. M. Whittaker,* and first solved by Valiron.† The result may be stated as follows:

THEOREM A. *Suppose that C is a simple closed curve which is a level curve of both the integral functions $f(z)$ and $g(z)$. Then either*

$$f(z) = K\{g(z)\}^{\alpha} \quad (\alpha \text{ real}), \tag{1}$$

or else there exists a number w_0 and an integral function $w(z)$ such that $|w(z)| = 1$ on C and $w(z)$ does not take the value w_0. In the latter case any integral function $f(z)$ for which C is a level curve can be expressed in the form

$$f(z) = K\{w(z)\}^{\mu}\left\{\frac{w(z)\bar{w}_0 - 1}{w(z) - w_0}\right\}^{\nu}, \tag{2}$$

where μ and ν are positive integers or 0.

2. The case of open level curves is not nearly so straightforward; although a branch of an open level curve on which $|f(z)| = M$ must contain the frontier of a simply-connected domain, it need not contain the frontier of a domain in which $|f(z)| < M$ nor of a domain in which $|f(z)| > M$. For example, each branch of the level curve $|\exp(\exp z)| = 1$ for which $y = (n+\frac{1}{2})\pi$, $z = x+iy$, divides the plane into two domains both containing points for which $|\exp(\exp z)| > 1$ and points for which $|\exp(\exp z)| < 1$. The functions e^{z} and $e^{-ie^{z}}$ have

* See K. Pennycuick (**4**).

† G. Valiron (**7**); see also M. L. Cartwright (**1**).

the imaginary axis as part of a common level curve, but cannot be expressed by any formula similar to (1); and more complicated examples of a similar type may be constructed with higher exponentials.

We shall not attempt to cover such cases; but we shall restrict ourselves to functions of finite order and to parts of level curves $|f(z)| = M$ which bound simply connected domains in which $\log\{|f(z)|/M\}$ does not change sign. As in the case of closed level curves, we obtain our results by considering meromorphic as well as integral functions: and the results obtained enable us to show that, *if C is a complete level curve for two integral functions $f(z)$ and $g(z)$ of finite order, then the result of Theorem A still holds.*

As a matter of fact, except in the case covered by Theorem A, either μ or ν must be 0 in (2).

3. In what follows we consider the whole complex plane including the point at infinity. In order to clarify our ideas we shall call the whole set of finite points at which $|f(z)| = M$ the *complete level curve* M. If the complete level curve is not connected, it can be divided into two or more components each of which is connected and closed relative to the complete level curve.* These components are called *branches* of the level curve. Any branch on which $|z|$ is bounded is closed; and the branches which extend to infinity are not closed. We therefore call the bounded branches *closed*, and the others *open* in accordance with the usual definitions.

In the following proofs we are chiefly concerned with branches or portions of branches which form the frontiers of certain domains. These are either simple closed curves or simple open curves tending to infinity in both directions. The latter can be expressed in the form $z = z(t)$, where $z(t)$ is continuous for $0 < t < 1$, $z(t_1) \neq z(t_2)$ unless $t_1 = t_2$, and $|z(t)| \to \infty$ as $t \to 0$ and as $t \to 1$. Any closed branch contains at least one simple closed curve; and any open branch contains at least one simple open curve tending to infinity in both directions. In the first three theorems we are mainly concerned with reducing the general problem for two functions having a complete level curve in common to certain simple cases in which parts of level curves bound simply connected domains in which $\log\{|f(z)|/M\}$ and $\log\{|g(z)|/M\}$ do not change sign.

* A set E contained in another set S is closed relative to S if all the limit points of E contained in S belong to E. E may have limit points which do not belong to S. See Newman **(3)**, 43 and 62.

THEOREM 1. *If the complete level curve M of the integral function $f(z)$ contains no closed branch as a sub-set, then the domains in which $|f(z)| > M$ are simply connected.*

Proof. Let D denote any domain in which $|f| > M$; and let $C(D)$ denote the complementary set to D including the point at infinity. Then $C(D)$ is connected. For, if not, it can be divided into two non-null non-intersecting sets F_1 and F_2 both closed in $C(D)$. Suppose that F_1 contains the point at infinity. Then F_2 does not, and, since $C(D)$ is closed, F_2 is closed; and so bounded. But F_2 is non-null, and therefore contains part, at least, of a branch of the level curve. The branch is connected; and so it lies entirely in F_2, and is bounded. Since $f(z)$ has no closed branches, we have a contradiction. Since $C(D)$ is connected, it follows that D is simply connected.

4. Every branch of an open level curve obviously contains a path of finite indetermination, and some theorems on the relation between level curves and asymptotic values have been given by Pennycuick* and Cartwright.† It is in this connexion that we use the hypothesis of finite order. The following theorem is an immediate corollary of a well-known result.‡

THEOREM 2. *If an integral function $f(z)$ is of finite order ρ, then the boundary of any domain D in which $|f(z)| > M$ contains at most a finite number $K(\rho)$ of non-contiguous paths of finite indetermination.*

Actually $K(\rho) = [2\rho]$, but we do not need this. We also need the following result.

THEOREM 3. *If the complete level curve M of an integral function $f(z)$ of finite order contains no simple closed curve, then there exists a domain D in which $\log(|f|/M)$ does not change sign such that the frontier $F(D)$ consists of a simple open curve tending to infinity in both directions.*

Proof. Consider any domain D in which $|f| > M$. Its frontier consists of one or more simple open curves. For, if the frontier of D_1 contains a multiple point, it contains a simple closed curve which is contrary to hypothesis. Suppose that $F(D_1)$ contains more than one simple open curve, and let C_1 be one of them. Then C_1 divides the finite part of the plane into two domains Δ_1 and Δ_1' such that Δ_1 contains D_1 and Δ_1' contains at least one domain d_1 in which $|f| < M$.

* Pennycuick (4). † Cartwright (1). ‡ Valiron (7), 145.

If $F(d_1)$ is the curve C_1, then d_1 is the required domain. If not, $F(d_1)$ contains at least one other simple open curve, c_1 say, which divides the plane into two domains δ_1 and δ_1' such that δ_1 contains Δ_1, C_1, and d_1, while δ_1' contains at least one domain D_2 in which $|f| > M$. Since $|f| > M$ in D_1 and D_2, it follows from the maximum-modulus principle that $|f|$ is unbounded in D_1 and D_2. The two directions in which C_1 tends to infinity are separated by Δ_1 and Δ_1', which include D_1 and D_2 respectively. Hence the two directions are separated by domains in which $|f|$ is unbounded; and so they form non-contiguous paths of indetermination.

If $F(D_2)$ is the curve c_1, then D_2 is the required domain. If not, $F(D_2)$ contains a curve C_2 which divides the plane into two domains Δ_2 and Δ_2' such that Δ_2 contains δ_1, c_1, and D_2, while Δ_2' contains at least one domain d_2 in which $|f| < M$. Repeating the argument we see that either d_2 is the required domain, or else $F(d_2)$ contains another curve c_2 dividing the plane into domains δ_2 and δ_2' such that δ_2 contains Δ_2, C_2, and d_2, while δ_2' contains a domain D_3 in which $|f|$ is unbounded.

The domain D_2 separates C_2 from C_1, and D_3 separates the two directions of C_2 from each other, so that we have four non-contiguous paths of determination. If $F(D_3)$ is the curve C_2, then D_3 is the required domain. If not, $F(D_3)$ contains another curve C_3, and we can repeat the process. Each time we repeat the argument we find either a domain of the required type, or a new path of finite indetermination. Since $f(z)$ is of finite order, the number of paths is finite; and so the process terminates in giving a domain of the required type.

5. We next prove a theorem about integral functions having common open level curves.

THEOREM 4. *Suppose that $f(z)$ and $g(z)$ are integral functions of finite order, and let D be a simply-connected domain such that $|f| > M$, $|g| > M'$ in D, and $|f| = M$, $|g| = M'$ on $F(D)$. Then*

$$f(z) = M\{g(z)/M'\}^{\alpha} \quad (\alpha > 0). \tag{3}$$

As in the case of Theorem A, we obtain this result by mapping D on the unit circle $|\zeta| < 1$; and we put $\phi(\zeta) = 1/f(z)$, so as to obtain a function which is *regular and bounded in* $|\zeta| < 1$. But here, since one point at least on $|\zeta| = 1$ corresponds to the point at infinity, $\phi(\zeta)$ has more complicated singularities on $|\zeta| = 1$. However, by the

well-known theorems* of Montel and Lindelöf, *if* $\phi(\zeta) \to \beta$ *as* $\zeta \to e^{i\alpha}$ *along any path in* $|\zeta| < 1$, *then* $\phi(\zeta) \to \beta$ *uniformly as* $\zeta \to e^{i\alpha}$ *in the angle* $|\arg(e^{i\alpha}-\zeta)| \leqslant \frac{1}{2}\pi-\delta$ *for every* $\delta > 0$, *and there is no path along which* $\phi(\zeta)$ *tends to any other limit as* $\zeta \to e^{i\alpha}$.

We shall standardize so that our function $\phi(\zeta)$ has modulus 1 at all points on $|\zeta| = 1$ except perhaps at an enumerable set corresponding to the point at infinity; and so it follows from Schwarz's reflection principle that $\phi(\zeta)$ can be continued across $|\zeta| = 1$ by the formula

$$\phi(\zeta)\overline{\phi(1/\bar{\zeta})} = 1. \tag{4}$$

We also need the following lemma which concerns a special case of functions which have been studied very thoroughly by Seidel.†

LEMMA 1. *Suppose that* $\phi(\zeta)$ *is regular and* $|\phi| < 1$ *for* $|\zeta| < 1$; *suppose further that* $|\phi| = 1$ *on* $|\zeta| = 1$ *except at an enumerable set of points* $e^{i\alpha_1}$, $e^{i\alpha_2}$,..., *and that* $\phi \neq 0$ *in* $|\zeta| < 1$. *Then*

$$\phi(\zeta) = \exp\left\{-\sum_{s=1}^{\infty} \sigma_s \frac{e^{i\alpha_s}+\zeta}{e^{i\alpha_s}-\zeta}\right\} \quad (\sigma_s > 0); \tag{5}$$

and $\phi \to 0$ *as* $\zeta \to e^{i\alpha_s}$ $(s = 1, 2,...)$.

Proof of Theorem 4. Since D is simply-connected, there is a function $\zeta(z)$ which maps D one-one and conformally on $|\zeta| < 1$. Let

$$\phi(\zeta) = 1/f(z), \qquad \psi(\zeta) = 1/g(z).$$

We may suppose without loss of generality that $M = M' = 1$. Since the finite part of $F(D)$ consists of pieces of continuous curves on which $|f| = |g| = 1$, those pieces correspond to open intervals $I_1, I_2,...$ on $|\zeta| = 1$ in which $|\phi| = |\psi| = 1$; the remaining points on $|\zeta| = 1$ correspond to the point at infinity, and are at most enumerable.

Now the number of points on $|\zeta| = 1$ at which either ϕ or ψ tends to 0 is finite. For, if not, given any integer N, we can find N points $e^{i\alpha_1}$, $e^{i\alpha_2}$,..., $e^{i\alpha_N}$, at which either $\phi(\zeta) \to 0$ or else $\psi(\zeta) \to 0$. Then there is an interval I_s between each consecutive pair of points, such that $|\phi| = |\psi| = 1$ in I_s, and $\beta_s < \arg\zeta < \beta'_s$, and

$$\alpha_1 \leqslant \beta_1 < \beta'_1 \leqslant \alpha_2 \leqslant \beta_2 < \beta'_2 < \ldots < \beta'_N \leqslant \alpha_1+2\pi.$$

Let C_s be the curve in the γ-plane corresponding to I_s; and let C'_s be the curve corresponding to the radius $(0, e^{i\alpha_s})$. Then $|f| \to \infty$ on C'_s, and $|f| = 1$ on C_s, so that C_s is a path of finite indetermination

* See Titchmarsh (6), 179 and 170. † Seidel (5).

tending to infinity as z tends to infinity in the directions corresponding to $e^{i\beta_s}$ and $e^{i\beta'_s}$. Each consecutive pair of curves C_s, C_{s+1} ($s = 1, 2, \ldots, N-1$) is separated by a curve C'_{s+1} on which $|f| \to \infty$, and C_N is separated from C_1 by C'_1. Hence we have N non-contiguous paths of finite indetermination; and, if N is sufficiently large, this contradicts Theorem 2. It follows from Lemma 1 that ϕ and ψ tend to 0 at all points other than those at which $|\phi| \to 1$ and $|\psi| \to 1$ respectively. For ϕ and ψ have no zeros in $|\zeta| < 1$, so that we may write

$$\phi(\zeta) = \exp\Big\{-\sum_{s=1}^{N} \frac{e^{i\alpha_s}+\zeta}{e^{i\alpha_s}-\zeta}\sigma_s\Big\}, \qquad \psi(\zeta) = \exp\Big\{-\sum_{s=1}^{N} \frac{e^{i\alpha_s}+\zeta}{e^{i\alpha_s}-\zeta}\tau_s\Big\}, \quad (6)$$

where $\sigma_s \geqslant 0$, $\tau_s \geqslant 0$.

It follows from (4) that ϕ and ψ are one-valued functions of ζ, and that $|\phi| > 1$ if and only if $|\psi| > 1$. Writing $\zeta = \phi^{-1}\{1/f(z)\}$, we see that the mapping function is regular everywhere both outside and inside D except perhaps at isolated points where $f(z) = 0$ or $f'(z) = 0$. Let D_1 be any domain in which $|f| < 1$ containing a zero z_1 of $f(z)$. Then $|g| < 1$ in D_1, and $g(z_1) = \gamma$, $|\gamma| < 1$. The domain D_1 is mapped on $|\zeta| > 1$ except perhaps for the zeros of $f(z)$ which correspond to points on $|\zeta| = 1$; but the mapping is not necessarily one-one and not necessarily conformal at points where $f(z) = 0$ or $f'(z) = 0$. As $z \to z_1$, $f(z) \to 0$, and so $|\phi| \to \infty$ at the corresponding point which must therefore be one of the points $e^{i\alpha_s}$, say $e^{i\alpha_1}$. Then as $z \to z_1$, $\zeta \to e^{i\alpha_1}$ from outside $|\zeta| = 1$. Hence $\psi \to 1/\gamma$ from outside, and $\psi \to \bar{\gamma}$ from inside. But, by Lemma 1, γ must be 0 or else $|\gamma| = 1$; and, since $|\gamma| < 1$, we have $\gamma = 0$; and so z_1 is a zero of $g(z)$. Hence there exist positive integers μ_1 and ν_1, and a number $k_1 \neq 0$ such that

$$F_1(z) = f^{\mu_1}/g^{\nu_1} \to k_1 \quad \text{as } z \to z_1;$$

and so

$$\Phi_1(z) = \phi^{\mu_1}/\psi^{\nu_1} \to 1/k_1 \quad \text{as } \zeta \to e^{i\alpha_1} \text{ from outside } |\zeta| = 1.$$

Let z_2 be any zero of $f(z)$ which does not correspond to $e^{i\alpha_1}$ and repeat the argument. If $e^{i\alpha_2}$ is the corresponding point on $|\zeta| = 1$, we have $\phi^{\mu_2}/\psi^{\nu_2} \to 1/k_2$ as $\zeta \to e^{i\alpha_2}$, where μ_2 and ν_2 are positive integers and $k_2 \neq 0$, and so on for $e^{i\alpha_s}$ ($s = 3, 4, \ldots, N$). We may suppose without real loss of generality that

$$\frac{\nu_1}{\mu_1} \leqslant \frac{\nu_s}{\mu_s}, \quad s > 1.$$

Then $F_1(z)$ is regular in D_1; and by (6)

$$\Phi_1(\zeta) = \exp\Big\{-\sum_{s=1}^{N} \frac{e^{i\alpha_s}+\zeta}{e^{i\alpha_s}-\zeta}(\mu_1\sigma_s-\nu_1\tau_s)\Big\}.$$

It is easy to see that $|\Phi_1|$ tends to 0, 1, or ∞ as $\zeta \to e^{i\alpha_s}$ according as $\mu_1\sigma_s-\nu_1\tau_s$ is greater than, equal to, or less than 0. Since $0<|k_1|<\infty$, and $|\Phi_1| \to |k_1|$ as $\zeta \to e^{i\alpha_1}$, we have $|k_1|=1$, and $\mu_1\sigma_1=\nu_1\tau_1$. The same argument applied to Φ_p gives $\mu_p\sigma_p=\nu_p\tau_p$ $(p=2,\ 3,\ldots,\ N)$; and, since $\nu_1/\mu_1 \leqslant \nu_p/\mu_p$, we have

$$\mu_1\sigma_p-\nu_1\tau_p = (\mu_1\nu_p-\mu_p\nu_1)\tau_p/\mu_p \geqslant 0 \quad (p=1,\ 2,\ldots,\ N).$$

Hence $|\Phi_1| \leqslant 1$ in $|\zeta| \leqslant 1$; and so by (4)

$$|F_1| = |1/\Phi_1| \leqslant 1$$

in D_1, and $|F_1| \to |k_1| = 1$ as $z \to z_1$ in D_1. It follows now from the maximum-modulus principle applied to $F_1(z)$ in D_1 that $F_1(z) = e^{i\lambda}$ $(\lambda \geqslant 0)$ in D_1. Hence $\{f(z)\}^{\mu_1} = e^{i\lambda}\{g(z)\}^{\nu_1}$ which is the required result for the case in which $f(z)$ has at least one zero.

If neither $f(z)$ nor $g(z)$ have any zeros, they are of the form

$$f(z) = \exp\Big\{\sum_1^N a_n z^n\Big\}, \qquad g(z) = \exp\Big\{\sum_1^N b_n z^n\Big\} \quad (a_N \neq 0,\ b_N \neq 0);$$

and the level curves lie near an adjacent pair of lines

$$\arg z^N = \pm\tfrac{1}{2}\pi - \arg a_N \quad \text{and} \quad \arg z^N = \pm\tfrac{1}{2}\pi - \arg b_N$$

respectively. Hence $\arg a_N = \pm \arg b_N$; and we can choose a real number α such that $a_N = \alpha b_N$. Then

$$F(z) = f(z)/\{g(z)\}^{\alpha} = \exp\Big\{\sum_1^{N-1}(a_n-\alpha b_n)z^n\Big\}$$

is also an integral function with the same level curves, which is impossible unless $F(z)$ is a constant. This gives (3).

6. It remains to deal with the case in which $|f|<M$, $|g|>M'$ in a domain bounded by parts of a complete level curve. This, as we shall see, can only happen when the functions have an exceptional value; and the result is similar to the second alternative in Theorem A. We can obviously restrict ourselves to the type of domain considered in Theorem 3; and, as in the proof of Theorem A, we consider meromorphic functions. The next two theorems are independent of the hypothesis of finite order.

THEOREM 5. *Let D be a domain bounded by a single simple open curve C. Suppose that $g(z)$ is an integral function such that $|g| > 1$ in D, and $g(z)$ takes the value 0 at least once. Suppose that $f(z)$ is a meromorphic function such that $|f| < 1$ in D and that $|f| = |g| = 1$ on C. Then $f(z)$ has one and only one asymptotic value β such that $|\beta| < 1$.*

Proof. Since $|f| < 1$ in D the domain is simply connected. This is also obvious from the fact that $C = F(D)$ is connected. Map D on $|\zeta| < 1$. Then the circumference $|\zeta| = 1$ corresponds to C except for one point which corresponds to the point at infinity; and we may choose the mapping function so that this point is $\zeta = 1$. Let

$$\phi(\zeta) = f(z), \qquad \psi(\zeta) = 1/g(z), \tag{6'}$$

so that ϕ and ψ are regular and $|\phi| < 1$, $|\psi| < 1$ for $|\zeta| < 1$. Further, $|\phi| = |\psi| = 1$ on $|\zeta| = 1$ except at $\zeta = 1$, so that ϕ and ψ can be continued across $|\zeta| = 1$ by the formula (4); and the domains in which $|f| > 1$ are those in which $|g| < 1$.

Since $\psi(z)$ has no zeros in $|\zeta| < 1$, it follows from Lemma 1 that

$$\psi(z) = \exp\left\{-\tau\frac{1+\zeta}{1-\zeta}\right\} \quad (\tau > 0). \tag{7}$$

Let z_0 be a zero of $g(z)$ (since there is at least one). Then as $z \to z_0$ $g(z) \to 0$ and $|\psi| \to \infty$. Hence $\zeta \to 1$ from outside the unit circle. Suppose that* $f(z_0) = 1/\bar{\beta}$, so that $\phi(\zeta) \to 1/\bar{\beta}$ as $\zeta \to 1$ from outside $|\zeta| = 1$. Then, by (4), $\phi(\zeta) \to \beta$ as $\zeta \to 1$ from inside $|\zeta| = 1$; and so $f(z) \to \beta$ along the corresponding curve in the z-plane. But, since $\zeta = 1$ corresponds to the point at infinity, this curve tends to infinity; and β is an asymptotic value. Further, since $|\phi| < 1$ in $|\zeta| < 1$, $\phi(\zeta)$ cannot tend to any other value as $\zeta \to 1$ from inside $|\zeta| = 1$; and all other points on $|\zeta| = 1$ correspond to finite points in the z-plane. Hence β is the only asymptotic value with modulus less than one.

THEOREM 6. *Suppose that $f(z)$ and $g(z)$ satisfy the hypotheses of Theorem 5, and let μ be the largest integer such that*

$$g_0(z) = \{g(z)\}^{1/\mu}$$

is an integral function. Then all the zeros of $g_0(z)$ are included in those of $\bar{\beta}f(z) - 1$.

We observe that, since $g(z)$ has at least one zero, there is an integer μ such that $\{g(z)\}^{1/m}$ is not an integral function for $m > \mu$.

* Since $f(z)$ is meromorphic, $\beta = 0$ is permissible if we interpret $f(z_0) = 1/\bar{\beta}$ by saying that z_0 is a pole of $f(z)$.

Proof. Using the notation of Theorem 5, we write

$$\phi(\zeta,\beta)=\frac{\phi(\zeta)-\beta}{\bar{\beta}\phi(\zeta)-1}, \qquad f(z,\beta)=\frac{f(z)-\beta}{\bar{\beta}f(z)-1}.$$

Then $|\phi(\zeta,\beta)|<1$ for $|\zeta|<1$, and $|\phi(\zeta,\beta)|=1$ on $|\zeta|=1$ except perhaps at $\zeta=1$. Suppose that $\bar{\beta}f(z)-1$ and $g(z)$ have zeros of the pth and qth orders respectively at z_0, and let $p=\nu_1 r$, $q=\mu_1 r$, where r is the highest common factor of p and q. Then both $\{\bar{\beta}f(z)-1\}^{\mu_1}$ and $\{g(z)\}^{\nu_1}$ have zeros of order $\mu_1\nu_1 r$ at z_0. Hence, as $z\to z_0$,

$$F(z)=\{f(z,\beta)\}^{\mu_1}\{g(z)\}^{\nu_1}\to\kappa \quad (0<|\kappa|<\infty); \tag{8}$$

and so, as $\zeta\to 1$ from outside $|\zeta|=1$,

$$\Phi(\zeta)=F(z)\to\kappa.$$

But, by (7), $\psi(\zeta)$ is finite except that, as $\zeta\to 1$ from outside $\zeta=1$, $|\psi|\to\infty$. Hence *every* zero of $g(z)$ corresponds to $\zeta=1$ approached from outside $|\zeta|=1$, and so (8) holds for all zeros of $g(z)$. Let z_1 be any other zero of $g(z)$. Then it is also a zero of $\bar{\beta}f(z)-1$; and, if q_1 and p_1 are the orders of the zeros of $g(z)$ and $\bar{\beta}f(z)-1$ respectively at z_1, we have $q_1\nu_1=p_1\mu_1$ so that μ_1 is a factor of q_1. Hence $g_1(z)=\{g(z)\}^{1/\mu_1}$ is an integral function; and all the zeros of $\{g_1(z)\}^{\nu_1}$ are included in those of $\bar{\beta}f(z)-1$. The required result follows *a fortiori*.

7. We need some notation and results from the Nevanlinna theory of meromorphic functions.* We write $n(r,f)$ for the number of zeros of $f(z)$ for which $|z|\leqslant r$,

$$m(r,f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+|f(re^{i\theta})|\,d\theta,$$

$$N(r,f)=\int_0^r\left\{n\left(u,\frac{1}{f}\right)-n\left(0,\frac{1}{f}\right)\right\}\frac{du}{u}+n\left(0,\frac{1}{f}\right)\log r,$$

$$T(r,f)=m(r,f)+N(r,f).$$

It is known that for any finite α

$$T\left(r,\frac{1}{f-\alpha}\right)=m\left(r,\frac{1}{f-\alpha}\right)+N\left(r,\frac{1}{f-\alpha}\right)=T(r,f)+O(1), \tag{9}$$

as $r\to\infty$; and $T(r,f)$ is an increasing convex function of r tending to infinity as $r\to\infty$.

* See Nevanlinna (**2**), 6.

THEOREM 7. *If $f(z)$ and $g(z)$ satisfy the hypotheses of Theorem 5, and if, in addition,*

$$\lim_{r\to\infty} \frac{T(r,f)}{T(r,g)} < \infty, \tag{10}$$

then there exist N positive integers ν_1, ν_2,..., ν_N, and N numbers β_1, β_2,..., β_N of modulus less than one such that

$$f_N(z) = e^{i\lambda} \quad (\lambda > 0), \tag{11}$$

where

$$\{g_0(z)\}^{\nu_s} f_s(z) = \frac{f_{s-1}(z)-\beta_s}{1-\bar{\beta}_s f_{s-1}(z)} \quad (s = 1, 2,..., N), \qquad f_0(z) = f(z), \tag{12}$$

and $g_0(z)$ is the function defined in Theorem 6.

We need another lemma.

LEMMA 2. *Suppose that*

(i) *$\phi(\zeta)$ is regular for $|\zeta| < 1$;*

(ii) *$\phi(\zeta)$ is regular and $|\phi| = 1$ on $|\zeta| = 1$ for $0 < |\arg \zeta| < \delta$;*

(iii) *$|\phi| \to \kappa$, $0 < |\kappa| < \infty$ as $\zeta \to 1$ along the real axis;*

(iv) $\log|\phi| \leqslant \tau \Re\left\{\dfrac{1+\zeta}{1-\zeta}\right\}$ $(\tau > 0)$ *for* $|\zeta| < 1$, $|1-\zeta| < \rho_0$.

Then $|\phi|$ is bounded for $|\zeta| < 1$, $|1-\zeta| < \rho_0$.

Proof. Put $Z = (1+\zeta)/(1-\zeta)$, so that the upper and lower semicircles of $|\zeta| = 1$ correspond to $\arg Z = \pm\frac{1}{2}\pi$ respectively, and the real axis corresponds to itself. Let $\chi(Z) = \phi(\zeta)$, and apply a theorem of Phragmèn and Lindelöf* to $\chi(Z)$ in the angle $0 < \arg Z < \frac{1}{2}\pi$. It follows from (ii) and (iii) that $\phi(\zeta)$ is bounded on the arms of the angle; and, by (iv),

$$|\chi| \leqslant \exp\{\tau|Z|\} \quad (|Z| > R_0).$$

Hence $|\chi|$ is bounded for $0 < \arg Z < \frac{1}{2}\pi$, $|Z| > R_0$, and similarly for $-\frac{1}{2}\pi < \arg Z < 0$; and so $|\chi|$ is bounded in the whole angle $-\frac{1}{2}\pi < \arg Z < \frac{1}{2}\pi$. The required result follows immediately.

Proof of Theorem 7. Put $g(z) = g_0(z)$ so that $\mu = 1$, and $\kappa = \kappa_1$, $\beta = \beta_1$ in the notation of Theorems 5 and 6; and let

$$f_1(z) = f(z,\beta_1)\{g_0(z)\}^{\nu_1}, \qquad \phi_1(\zeta) = f_1(z).$$

* See Titchmarsh (**6**), 177.

Then $f_1(z)$ is a meromorphic function. Since $g_0(z)$ has no poles, and the zeros of $g_0^{\nu_1}$ are included in those of $\bar{\beta}_1 f-1$, we have

$$N(r,f_1) = N\{r,f(z,\beta_1)\}-N\left(r,\frac{1}{g_0^{\nu_1}}\right)$$

$$= N\left(r,\frac{1}{f-1/\bar{\beta}_1}\right)-\nu_1 N\left(r,\frac{1}{g_0}\right).$$

By (8), $f_1(z)\to\kappa_1$ as $z\to z_0$; and so

$$\phi_1\to 1/\bar{\kappa}_1 \quad (0<|\kappa_1|<\infty), \tag{13}$$

as $\zeta\to 1$ along the real axis from inside $|\zeta|=1$. Further, ϕ_1 obviously satisfies (i), (ii), (iii) of Lemma 2, and

$$\log|\phi_1| \leqslant -\nu_1\log|\psi| \leqslant \nu_1\tau\,\Re\left(\frac{1+\zeta}{1-\zeta}\right) \quad (|\zeta|<1),$$

so that (iv) holds. Hence $|\phi_1|$ is bounded for $|\zeta|<1$; and, since $|\phi_1|\to 1$ as $|\zeta|\to 1$ except perhaps at $\zeta=1$, by the maximum-modulus principle we have $|\phi_1|\leqslant 1$ for $|\zeta|<1$. It now follows from Lemma 1 that $|\phi_1|\to 1$ as $\zeta\to 1$. For, if not, $|\phi_1|\to 0$, which is contradicted by (13). Now, if at any point in the z-plane, $|g_0(z)|>1$, then at the corresponding point in the ζ-plane $|\zeta|<1$; and so $|\phi_1(\zeta)|<1$, and $|f_1(z)|<1$ at these points. Hence there is no contribution to $m(r,f_1)$ from points at which $|g_0|>1$; and so, if $|f_1|>1$, we must have $|g_0|<1$, and $|f(z,\beta_1)|>1$. Hence

$$m(r,f_1) = m\{r,f(z,\beta_1)\}-m\left(r,\frac{1}{g^{\nu_1}}\right).$$

Further, since $|f(z,\beta_1)|<K(\beta_1)$ for $|f(z)|>K'(\beta_1)$, and

$$f(z,\beta_1) = \frac{1}{\bar{\beta}_1}+\frac{1-|\beta_1|^2}{\bar{\beta}_1^2}\frac{1}{f(z)-1/\bar{\beta}_1},$$

$$\log|f(z,\beta_1)| = \log\frac{1}{|f-1/\bar{\beta}_1|}+\log\frac{1-|\beta_1|^2}{|\beta_1|^2}+\log\left|1+\frac{\bar{\beta}_1 f-1}{1-|\beta_1|^2}\right|,$$

we have $$m\{r,f(z,\beta_1)\} \leqslant m\left(r,\frac{1}{f-1/\bar{\beta}_1}\right)+K_1(\beta_1).$$

Hence $T(r,f_1) = m(r,f_1)+N(r,f_1)$

$$\leqslant m\left(r,\frac{1}{f-1/\bar{\beta}_1}\right)-\nu_1 m\left(r,\frac{1}{g_0}\right)+N\left(r,\frac{1}{f-1/\bar{\beta}_1}\right)-\nu_1 N\left(r,\frac{1}{g_0}\right)+K_1(\beta_1)$$

$$= T\left(r,\frac{1}{f-1/\bar{\beta}_1}\right)-\nu_1 T\left(r,\frac{1}{g_0}\right)+K_1(\beta_1)$$

$$\leqslant T(r,f)-\nu_1 T(r,g_0)+K_2(\beta_1).$$

If $f_1(z)$ is not a constant, we can repeat the argument with f_1 in place of f; and we have

$$T(r,f_2) \leqslant T(r,f)-(\nu_1+\nu_2)T(r,g_0)+\kappa_3(\beta_1,\beta_2),$$

where $\beta_2 = 1/\bar{\kappa}_1$, and ν_2 and f_2 are obtained from f_1 and g_0 just as ν_1 and f_1 were obtained from f and g_0. Since $\nu_1, \nu_2, \ldots$ are integers, and since (10) holds, if we repeat the process a finite number of times, we obtain a function $f_N(z)$ such that

$$\overline{\lim_{r\to\infty}}\, T(r,f_N) < \infty,$$

which is impossible unless $f_N(z)$ is a constant; and since $|f_N| = 1$ on C, we have $f_N = e^{i\lambda}$ $(\lambda > 0)$, which is the result stated.

8. We now return to the special case in which $f(z)$ is an integral function of finite order.

THEOREM 8. *If $f(z)$ and $g(z)$ satisfy the hypotheses of Theorem 5, and if $f(z)$ is an integral function of finite order, then* (10) *holds with upper limit in place of lower limit.*

Proof. Let Δ be any domain in which $|f| > 1$. As we have already seen, if $|f| > 1$ in Δ, then $|g| < 1$; and, since $g(z)$ has no poles, Δ is simply connected. Hence we can map Δ on $|\zeta| < 1$. As in Theorem 4, we have

$$\phi(\zeta) = \frac{1}{f(z)} = \exp\left\{-\sum_{s=1}^{N} \sigma_s \frac{e^{i\alpha_s}+\zeta}{e^{i\alpha_s}-\zeta}\right\} \quad (\sigma_s \geqslant 0), \quad \text{for } |\zeta| < 1.$$

We may suppose that zero is not an asymptotic value of $f(z)$. For, by Theorem 5, $f(z)$ has only one asymptotic value β_1 of modulus less than one; and, if $\beta_1 = 0$, it will be sufficient to prove the result for $f_1(z) = f(z)\{g_0(z)\}^{\nu_1}$, which has an asymptotic value $\beta_2 = 1/\bar{\kappa}_1 \neq 0$.

As $\zeta \to e^{i\alpha_s}$ from outside $|\zeta| = 1$, $|\phi(\zeta)| \to \infty$; and so $f(z) \to 0$ along the corresponding curve. Since zero is not an asymptotic value, z tends to z_s a zero of $f(z)$. Let $g(z_s) = \gamma_s$. Then we can choose integers m_s and n_s such that

$$g_s(z) = \left(\frac{g-\gamma_s}{1-\bar{\gamma}_s g}\right)^{n_s} \Big/ \{f(z)\}^{m_s} \to k_s \quad (0 < |k_s| < \infty),$$

as $z \to z_1$. Hence $\qquad \psi_s(\zeta) = g_s(z) \to 1/\overline{k}_s$

as $\zeta \to e^{i\alpha_s}$ from inside $|\zeta| = 1$. By Lemma 3, $\psi_s(\zeta)$ is bounded near $e^{i\alpha_s}$ for $|\zeta| < 1$; and so the contribution to $m(r,f)$ from points in the z-plane corresponding to points near $e^{i\alpha_s}$ is less than $K_s m(r,g)$. Summing over $s = 1, 2, \ldots, N$, and remembering that, since $f(z)$ is an

integral function, $T(r,f) = m(r,f)$, we have $T(r,f) < KT(r,g)$ which includes (10) and gives the required result.

9. We now have to extract the special result for integral functions from Theorems 7 and 8.

THEOREM 9. *Suppose that $f(z)$ and $g(z)$ satisfy the hypotheses of Theorem 8. Then there exist positive integers μ and ν and a number γ such that*

$$g(z) = \{g_0(z)\}^\mu, \qquad f(z) = e^{i\lambda}\left\{\frac{1-\bar{\gamma}g_0(z)}{g_0-\gamma}\right\}^\nu,$$

where $\lambda > 0$ and $g_0(z)$ is an integral function which does not take the value γ.

Proof. It follows from Theorems 7 and 8 that (11) and (12) hold. Hence

$$f(z) = \frac{\beta_1+g_0^{\nu_1}f_1}{1+\bar{\beta}_1 g_0^{\nu_1}f_1} = \frac{\beta_1+\beta_1\bar{\beta}_2 g_0^{\nu_2}f_2+\beta_2 g_0^{\nu_1}+g_0^{\nu_1+\nu_2}f_2}{1+\bar{\beta}_2 g_0^{\nu_2}f_2+\bar{\beta}_1\beta_2 g_0^{\nu_1}+\bar{\beta}_1 g_0^{\nu_1+\nu_2}f_2},$$

and so on until we reach $f_N = e^{i\lambda}$. Hence

$$f(z) = \frac{\sum_{n=0}^{M} a_n g_0^n}{\sum_{n=0}^{M} b_n g_0^n} \quad (M = \nu_1+\nu_2+\ldots+\nu_N), \tag{14}$$

where $a_M = e^{i\lambda}$, $b_0 = 1$, and generally a_n, b_n are functions of $\beta_1, \beta_2, \ldots, \beta_N$ and $e^{i\lambda}$.

Since $f(z)$ is an integral function, and since the numerator and denominator have no common factors,* the denominator cannot vanish. We may obviously suppose that $b_{M'} \neq 0$ $(0 \leqslant M' \leqslant M)$ and $b_n = 0$ $(n > M')$. Then there are M' values of g_0, some of which may coincide, for which

$$\sum_{n=0}^{M} b_n g_0^n = \sum_{n=0}^{M'} b_n g_0^n = 0. \tag{15}$$

Hence g_0 omits these M' values; but, since g_0 is an integral function, it cannot omit more than one value $1/\bar{\gamma}$, say; and so the M' values must coincide. Hence, remembering that $b_0 = 1$, we have

$$\sum_{n=0}^{M'} b_n g_0^n = (1-\bar{\gamma}g_0)^{M'}.$$

Suppose that g_0 takes a certain value G at z_1 and the value $1/\bar{G}$ at z'. Then, by (4) and (6′),

$$\overline{\zeta(z_1')}\zeta(z_1) = 1;$$

and so

$$\overline{f(z_1')}f(z_1) = 1.$$

* This is obvious from the formula connecting f and f_1.

By (14),
$$\overline{f(z_1')} = \frac{\sum \overline{a_n} G^{-n}}{\sum \overline{b_n} G^{-n}},$$

so that
$$\frac{\sum \overline{a_n} G^{M-n}}{\sum \overline{b_n} G^{M-n}} = \frac{\sum b_n G^n}{\sum a_n G^n},$$

where $a_M = e^{i\lambda}$ and $b_0 = 1$. This is true for all values G; and so it is easy to see that
$$\sum \overline{a_n} G^{M-n} \equiv e^{-i\lambda} \sum b_n G^n.$$

For both expressions have the same zeros considered as polynomials in G; and, since $\overline{a_M} = e^{-i\lambda}$, $b_0 = 1$, they are identical. Hence
$$f(z) = e^{i\lambda}\left(\frac{g_0 - \gamma}{1 - \bar{\gamma} g_0}\right)^M,$$

which is equivalent to the required result.

10. It remains to apply our results to integral functions having a complete level curve in common.

THEOREM 10. *If $f(z)$ and $g(z)$ are integral functions of finite order having the complete level curve C on which $|f| = M$, $|g| = M'$ in common, then the result of Theorem A holds, and further in* (2) *either $\mu = 0$ or $\nu = 0$.*

Proof. If C contains a simple closed curve, Theorem A itself covers the result.

If not, consider any domain D in which $|f| > M$, then either $|g| > M'$, in which case the result follows from Theorems 1 and 4, or else $|g| < M'$. If one or other of f and g omits the value zero, we can obviously reduce the second case to the first. If not, by Theorem 3, we can reduce it to the case of Theorem 9. In all cases we obtain the required result.

REFERENCES

1. M. L. Cartwright, 'On the level curves of integral and meromorphic functions': *Proc. London Math. Soc.* (2) 43 (1937), 468–74.
2. R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes* (Paris, 1929).
3. M. H. A. Newman, *Elements of the Topology of Plane Sets of Points* (Cambridge, 1939).
4. K. Pennycuick, 'On level curves of integral functions': *J. of London Math. Soc.* 11 (1936), 281–5.
5. W. Seidel, 'On the distribution of values of bounded analytic functions': *Trans. American Math. Soc.* 36 (1934), 201–26.
6. E. C. Titchmarsh, *The Theory of Functions*, 2nd edition (Oxford, 1939).
7. G. Valiron, *Lectures on the General Theory of Integral Functions* (Toulouse, 1923).
8. —— 'Sur les courbes de module constant des fonctions entières': *Comptes Rendus*, 204 (1932), 402–4.

ON THE ESTIMATION OF $N(\sigma, T)$

By A. E. INGHAM (*Cambridge*)

[Received 10 October 1940]

LET $N(\sigma, T)$ be the number of zeros $\rho = \beta+\gamma i$ of the Riemann zeta-function $\zeta(s)$ for which $\beta \geqslant \sigma$, $0 < \gamma \leqslant T$. It is proved in my paper [2] that the estimate

$$N(\sigma, T) = O(T^{\lambda(\sigma)(1-\sigma)}\log^5 T) \tag{1}$$

holds uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1$ as $T \to \infty$, when

$$\lambda(\sigma) = 1+2\sigma, \tag{2}$$

and also when

$$\lambda(\sigma) = 2+4c, \tag{3}$$

where c is a constant for which

$$\zeta(\tfrac{1}{2}+ti) = O(t^c) \quad \text{as } t \to \infty.$$

In this note we sketch a proof of (1) with

$$\lambda(\sigma) = \frac{3}{2-\sigma}.$$

This is an improvement on (2), since

$$(1+2\sigma)-\frac{3}{2-\sigma} = \frac{(2\sigma-1)(1-\sigma)}{2-\sigma} > 0 \quad (\tfrac{1}{2} < \sigma < 1),$$

but does not help in the application to the problem of the order of magnitude of the difference between consecutive primes, because it is the maximum of $\lambda(\sigma)$ for $\frac{1}{2} \leqslant \sigma \leqslant 1$ that is important in this problem.

We assume that the reader is familiar with § 3 and § 5 of [2], and merely indicate the essential changes in the argument.

As in [2], let

$$f_X(s) = \zeta(s)\sum_{n<X}\mu(n)n^{-s}-1 = \zeta(s)M_X(s)-1.$$

For $\sigma = \sigma_1 = \frac{1}{2}$, $\mu = \mu_1 = \frac{4}{3}$, we have, using (26) and a formula near the bottom of page 260 of [2],

$$\begin{aligned}
\int_0^T |f_X(\sigma+ti)|^\mu\, dt &\leqslant \int_0^T A_1(|\zeta|^{\frac{1}{3}}|M_X|^{\frac{1}{3}}+1)\, dt \\
&\leqslant A_1\left(\int_0^T |\zeta|^4\, dt\right)^{\frac{1}{3}}\left(\int_0^T |M_X|^2\, dt\right)^{\frac{2}{3}}+A_1 T \\
&< A_2\{T\log^4(T+2)\}^{\frac{1}{3}}\{(T+X)\log X\}^{\frac{2}{3}}+A_1 T \\
&< A_3(T+X)\log^2(T+X) \\
&< A_4(T+X)^{1+\delta}\delta^{-2} \quad (T > 0,\ X \geqslant 3,\ \delta > 0),
\end{aligned}$$

where A_1, A_2,... are positive absolute constants.

For $\sigma = \sigma_2 = 1+\delta$ $(0 < \delta < 1)$, $\mu = \mu_2 = 2$, we have, by [2], (17),

$$\int_0^T |f_X(\sigma+ti)|^\mu \, dt < A_5(T+X)X^{-1}\delta^{-4} \quad (T > 0,\ X \geqslant 3).$$

Arguing substantially as on page 261 of [2], but using a two-variable convexity theorem of Gabriel [1, Theorem 2]*, we deduce that, if $\frac{1}{2} \leqslant \sigma \leqslant 1+\delta$ $(0 < \delta < 1)$, and μ, ρ_1, ρ_2 are defined by

$$\frac{1}{\mu} = \frac{\vartheta_1}{\mu_1} + \frac{\vartheta_2}{\mu_2}, \qquad \rho_1 = \frac{\vartheta_1 \mu}{\mu_1}, \qquad \rho_2 = \frac{\vartheta_2 \mu}{\mu_2},$$

where

$$\vartheta_1 = \frac{1+\delta-\sigma}{\frac{1}{2}+\delta}, \qquad \vartheta_2 = \frac{\sigma-\frac{1}{2}}{\frac{1}{2}+\delta},$$

then, for $T > 1$, $X \geqslant 3$,

$$\int_1^T |f_X(\sigma+ti)|^\mu \, dt < A_6\{(T+X)^{1+\delta}\delta^{-2}\}^{\rho_1}\{(T+X)X^{-1}\delta^{-4}\}^{\rho_2}. \tag{4}$$

Now $\rho_1+\rho_2 = 1$,

$$\frac{1}{\mu} = \frac{\frac{3}{4}(1+\delta-\sigma)}{\frac{1}{2}+\delta} + \frac{\frac{1}{2}(\sigma-\frac{1}{2})}{\frac{1}{2}+\delta} = \frac{1+\frac{3}{2}\delta-\frac{1}{2}\sigma}{1+2\delta},$$

$$\rho_2 = \frac{(\sigma-\frac{1}{2})(1+2\delta)}{(\frac{1}{2}+\delta)(2+3\delta-\sigma)} = \frac{2\sigma-1}{2+3\delta-\sigma} > \frac{2\sigma-1}{2-\sigma} - A_7\delta.$$

Substituting in (4) and taking $\delta = A_8/\log(T+X)$, we obtain

$$\int_1^T |f_X(\sigma+ti)|^\mu \, dt < A_9(T+X)X^{-(2\sigma-1)/(2-\sigma)}\log^4(T+X),$$

for $T > 1$, $X \geqslant 3$, $\frac{1}{2} \leqslant \sigma \leqslant 1$, and a certain $\mu = \mu(\sigma, T, X)$ in the range $\frac{4}{3} \leqslant \mu \leqslant 2$.

Arguing now as on pages 262–3 of [2], using the fact that

$$\log|1-f_X^2| \leqslant \log(1+|f_X|^2) \leqslant A_{10}|f_X|^\mu \quad (\tfrac{4}{3} \leqslant \mu \leqslant 2),$$

and taking $X = T$, we obtain the stated result.

A similar argument with $\mu_1 = 1$, and with the mean value of $|\zeta(\frac{1}{2}+ti)|^2$ in place of that of $|\zeta(\frac{1}{2}+ti)|^4$, would give Titchmarsh's value $\lambda(\sigma) = 4/(3-2\sigma)$.

* The theorem is applied, with $\alpha = \sigma_1$ $(= \frac{1}{2})$, $\beta = \sigma_2$ $(= 1+\delta)$, $a = 1/\mu_1$, $b = 1/\mu_2$, $\lambda = \vartheta_1$, $\lambda' = \vartheta_2$, to the auxiliary function $\phi(s) = \phi_{X,T}(s)$ constructed in [2], 261.

REFERENCES

1. R. M. Gabriel, 'Some results concerning the integrals of moduli of regular functions along certain curves': *J. of London Math. Soc.* 2 (1927), 112–17.
2. A. E. Ingham, 'On the difference between consecutive primes': *Quart. J. of Math.* (Oxford), 8 (1937), 255–66.

ON FRACTIONAL INTEGRATION AND ITS APPLICATION TO THE THEORY OF HANKEL TRANSFORMS

By A. ERDÉLYI (*Edinburgh*)

[Received 10 September 1940]

1. Introduction

In a recent paper* H. Kober dealt with certain operators of fractional integration, $I^{\pm}_{\eta,\alpha}$ and $K^{\pm}_{\eta,\alpha}$, and in a subsequent paper† by Dr. Kober and myself some applications of these operators to the theory of Hankel transforms were given. Kober's operators were defined in a half-plane of the complex parameter η. $K^{-}_{\eta,\alpha}$, for instance, when operating on a function belonging to $L_p(0,\infty)$, is only defined for $\Re(\eta) > -1/p$. This restriction on η brought about a corresponding restriction on the validity of our theorems in E-K regarding the connexion between Hankel transforms of different order.

In what follows it will be pointed out that, by 'cutting' the kernel of the operators $I^{\pm}_{\eta,\alpha}$ and $K^{\pm}_{\eta,\alpha}$ when necessary, the definition of these operators can be extended to the greater part of the plane of the complex parameter η. As in all similar cases (cf., for instance, the ordinary and cut Hankel transform), the extension will not hold in the whole complex η-plane. There will be an infinity of equidistant parallel lines—in the case of $K^{-}_{\eta,\alpha}$ the lines $\Re(\eta) = -h-1/p$ ($h = 0, 1, 2,\ldots$)—where the operators remain undefined. It seems that on these lines the operators cannot be defined so as to be applicable to *any* function of $L_p(0,\infty)$.

The first part of the present paper contains those of the fundamental theorems on the operators $I^{+}_{\eta,\alpha}$ and $K^{-}_{\eta,\alpha}$ which are needed for Part II. In Part I functions belonging to classes $L_p(0,\infty)$, $(1 \leqslant p \leqslant \infty)$ have been considered.

In the second part of the paper, which brings the applications of our operators to the theory of Hankel transforms, I restricted myself for the sake of simplicity to functions of the class $L_2(0,\infty)$. Although from E-K it is seen that similar theorems hold for other L-classes,

* See above pp. 193–211. This paper will be quoted as K, so that K (7.6) or K Theorem 2 means equation (7.6) or Theorem 2 respectively of Kober's paper on fractional integration.

† See above pp. 212–21. This paper will be quoted as E-K.

it is obvious that the results will be particularly simple for L_2, which is its own conjugate class.

The proofs of the theorems of both parts of the present paper are so similar to the proofs of the corresponding theorems of K and E-K respectively that I decided to omit the proofs. Only in a few cases have I given some indication of the necessary changes in the proofs of the corresponding theorems on the restricted operators.

In the restricted range of definition, the operators defined here being identical with the old ones, I denote the extended operations by the same symbol as the restricted ones.

The norm of a function belonging to $L_p(a,b)$ will be written

$$|f|_p = \left\{\int_a^b |f(t)|^p\, dt\right\}^{1/p}$$

when $1 \leqslant p < \infty$, and $|f|_p =$ essential upper bound of $f(t)$ in the closed interval (a,b) when $p = \infty$. $L_p(0,\infty)$ will be written simply L_p.

By A some constant will be denoted, depending on the parameters concerned, not necessarily the same even when this symbol occurs several times in the same theorem.

I. Fractional Integrals and Derivatives

2. Definitions

Throughout the first part $1 \leqslant p \leqslant \infty$, $p' = p/(p-1)$, and $\Re(\alpha) > 0$; h and k are non-negative integers: in particular

$$h = 0 \text{ when } \Re(\eta) > -\frac{1}{p'},$$

and

$$-\frac{1}{p'} < \Re(\eta)+h < \frac{1}{p} \text{ when } \Re(\eta) < -\frac{1}{p'};$$

$$k = 0 \text{ when } \Re(\zeta) > -\frac{1}{p},$$

and

$$-\frac{1}{p} < \Re(\zeta)+k < \frac{1}{p'} \text{ when } \Re(\zeta) < -\frac{1}{p}.$$

Empty sums are to be interpreted as zero. Negative integer values of $\Re(\eta)-1/p$ and $\Re(\zeta)-1/p'$ should be excluded throughout.

The definitions of the operators are

$$I^+_{\eta,\alpha}\{f(x)\} = \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)}\left\{\int_0^x \left[(x-u)^{\alpha-1} - \sum_{r=0}^{h-1}\binom{\alpha-1}{r}x^{\alpha-r-1}(-u)^r\right]u^\eta f(u)\,du - \right.$$

$$\left. - \int_x^\infty \left[\sum_{r=0}^{h-1}\binom{\alpha-1}{r}x^{\alpha-r-1}(-u)^r\right]u^\eta f(u)\,du\right\}, \quad (1)$$

$$K^-_{\zeta,\alpha}\{f(x)\} = \frac{x^\zeta}{\Gamma(\alpha)}\left\{-\int_0^x \left[\sum_{s=0}^{k-1}\binom{\alpha-1}{s}u^{\alpha-s-1}(-x)^s\right]u^{-\zeta-\alpha}f(u)\,du + \right.$$

$$\left. + \int_x^\infty \left[(u-x)^{\alpha-1} - \sum_{s=0}^{k-1}\binom{\alpha-1}{s}u^{\alpha-s-1}(-x)^s\right]u^{-\zeta-\alpha}f(u)\,du\right\}, \quad (2)$$

$$I^-_{-\zeta-\alpha,\alpha}f = K^-_{\zeta,\alpha}f, \quad (3)$$

$$K^+_{-\eta-\alpha,\alpha}f = I^+_{\eta,\alpha}f. \quad (4)$$

In what follows we shall generally use only the operators $I^+_{\eta,\alpha}$ and $K^-_{\zeta,\alpha}$. It is to be understood that whenever operators with other parameters appear, for instance $I^+_{\eta+\alpha,\beta}$ and $K^-_{\epsilon,\alpha}$, h and k are to be replaced by the corresponding integers associated with $\eta+\alpha$ and ϵ.

The meaning of $I^+_{\eta,\alpha}$, for instance, can be made clear in the following way: when $\Re(\eta) < -1/p'$, then $u^\eta f(u)$ in general, i.e. with an arbitrary function $f(u)$ belonging to L_p, is not integrable in any interval $(0, x)$. Hence we first integrate h times from ∞ to x, thus getting a function which can be integrated [$(\alpha-h)$ times] from 0 to x. So (1) is equivalent, provided that $\Re(\alpha) > h$, to

$$I^+_{\eta,\alpha}f = \frac{x^{-\eta-\alpha}}{\Gamma(\alpha-h)}\int_0^x du\,(x-u)^{\alpha-h-1}\frac{(-)^h}{\Gamma(h)}\int_u^\infty (v-u)^{h-1}v^\eta f(v)\,dv. \quad (5)$$

Thus $x^{\eta+\alpha}I^+_{\eta,\alpha}f$ may be called the 'Riemann-Liouville integral of order α of $x^\eta f(x)$', although it would seem inappropriate to add 'with origin zero'.

Similarly, $x^{-\zeta}K^-_{\zeta,\alpha}f$ may be called the 'Weyl integral of order α of $x^{-\zeta-\alpha}f(x)$', the correct meaning of this being that as long as, with an arbitrary function of L_p, it would be impossible to integrate from

x to ∞ (i.e. k times), we integrate from x to zero, and then $(\alpha-k)$ times from x to infinity.

3. Particular cases

When $\Re(\eta) > -1/p'$ or $\Re(\zeta) > -1/p$, then the sums are empty and our operators reduce to Kober's definition

$$I^+_{\eta,\alpha}f = \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)}\int_0^x (x-u)^{\alpha-1}u^\eta f(u)\,du \quad \left(\Re(\eta) > -\frac{1}{p'}\right), \tag{6}$$

$$K^-_{\zeta,\alpha}f = \frac{x^\zeta}{\Gamma(\alpha)}\int_x^\infty (u-x)^{\alpha-1}u^{-\zeta-\alpha}f(u)\,du \quad \left(\Re(\zeta) > -\frac{1}{p}\right). \tag{7}$$

The other extreme case occurs when α is equal to a positive integer, not exceeding h and k respectively. In these cases the first term in (1) and the second term in (2) respectively vanish and we have

$$I^+_{\eta,m}f = -\frac{x^{-\eta-m}}{\Gamma(m)}\int_x^\infty (x-u)^{m-1}u^\eta f(u)\,du = (-)^m I^-_{\eta,m}f \qquad (m = 1, 2,\ldots, h), \tag{8}$$

and

$$K^-_{\zeta,m}f = -\frac{x^\zeta}{\Gamma(m)}\int_0^x (u-x)^{m-1}u^{-\zeta-\alpha}f(u)\,du = (-)^m K^+_{\zeta,m}f \qquad (m = 1, 2,\ldots, k). \tag{9}$$

4. Existence of the operations

THEOREM 1. *Let $f(x)$ belong to L_p. Then $I^+_{\eta,\alpha}f$ and $K^-_{\zeta,\alpha}f$ exist almost everywhere in $(0,\infty)$; also*

$$|I^+_{\eta,\alpha}f|_p \leqslant A|f|_p \quad \textit{and} \quad |K^-_{\zeta,\alpha}f|_p \leqslant A|f|_p.$$

This theorem corresponds to* K Theorem 2 with conditions (i). The proof is similar to that in K and is based on an inequality of Hardy-Littlewood-Pólya-Schur† slightly generalized by Kober.‡ The only difference compared with K is that Lemma 1 of K has to

* Cf. also G. H. Hardy and J. E. Littlewood, *Math. Zeits.* 27 (1928), 565–606, Theorems 7–11.

† I. Schur, *J. für Math.* 140 (1911), 1–28, for $p = 2$, G. H. Hardy, J. E. Littlewood, and G. Pólya, *Proc. London Math. Soc.* (2) 25 (1926), 265–82, for $1 < p < \infty$. Cf. also (by the same authors) *Inequalities* (Cambridge, 1934), Theorem 319.

‡ K Lemma 1.

be applied to the function

$$H(u,x) = u^{\eta}x^{-\eta-\alpha}\left\{(x-u)^{\alpha-1} - \sum_{r=0}^{h-1}\binom{\alpha-1}{r}x^{\alpha-r-1}(-u)^r\right\} \quad (0 \leqslant u < x),$$

$$H(u,x) = -u^{\eta}x^{-\eta-\alpha}\sum_{r=0}^{h-1}\binom{\alpha-1}{r}x^{\alpha-r-1}(-u)^r \qquad (x < u),$$

when dealing with $I^+_{\eta,\alpha}$, and to a corresponding function when treating $K^-_{\zeta,\alpha}$.

Now (i) $H(u,x)$ is homogeneous in u and x of degree -1;

(ii) $H(u,1)$ is integrable in any finite interval $0 < u_0 \leqslant u \leqslant u_1$; also

$$H(u,1) = O(u^{\eta+h}) \quad (u \to 0) \quad \text{and} \quad H(u,1) = O(u^{\eta+h-1}) \quad (u \to \infty).$$

Hence, since $-1/p' < \Re(\eta)+h < 1/p$,

$$\int_0^{\infty} |H(u,1)|u^{-1/p}\,du$$

exists. So K Lemma 1 can be applied in this case also and the rest of the proof is very similar to that in K § 2.

K (3.2) and K (3.3) also hold for the extended operators.

5. Fractional integration by parts

THEOREM 2. *Let $f(x)$ and $\phi(x)$ belong to L_p and $L_{p'}$ respectively. Then*

$$\int_0^{\infty} dx\,\phi(x)I^+_{\eta,\alpha}f(x) = \int_0^{\infty} dx\,f(x)K^-_{\eta,\alpha}\phi(x).$$

This corresponds to K Theorem 3 with conditions (i). The proof is similar to that in K § 3. In

$$\int_0^{\infty} dx\,\phi I^+_{\eta,\alpha}f = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} dx\,\phi(x)x^{-\eta-\alpha}\times$$

$$\times\int_0^{x}\left[(x-u)^{\alpha-1} - \sum_{r=0}^{h-1}\binom{\alpha-1}{r}x^{\alpha-r-1}(-u)^r\right]u^{\eta}f(u)\,du -$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^{\infty} dx\,\phi(x)x^{-\eta-\alpha}\int_x^{\infty}\left[\sum_{r=0}^{h-1}\binom{\alpha-1}{r}x^{\alpha-r-1}(-u)^r\right]u^{\eta}f(u)\,du$$

the order of integrations in both repeated integrals can be changed because of the absolute convergence. Hence we get

$$\frac{1}{\Gamma(\alpha)}\int_0^\infty du\, f(u)u^\eta \int_u^\infty \left[(x-u)^{\alpha-1} - \sum_{r=0}^{h-1}\binom{\alpha-1}{r}x^{\alpha-r-1}(-u)^r\right]x^{-\eta-\alpha}\phi(x)\,dx -$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^\infty du\, f(u)u^\eta \int_0^u \left[\sum_{r=0}^{h-1}\binom{\alpha-1}{r}x^{\alpha-r-1}(-u)^r\right]x^{-\eta-\alpha}\phi(x)\,dx$$

$$= \int_0^\infty du\, f(u)K^-_{\eta,\alpha}\phi(u).$$

This proves the theorem.

6. Repeated fractional integration

THEOREM 3. *Let $\Re(\alpha) > 0$ and $\Re(\beta) > 0$, and let $f(x)$ belong to L_p. Suppose that neither $\Re(\eta+\alpha)-1/p$ nor $\Re(\eta+\beta)-1/p$ is a negative integer. Then*

$$I^+_{\eta,\alpha}(I^+_{\eta+\alpha,\beta}f) = I^+_{\eta+\alpha,\beta}(I^+_{\eta,\alpha}f) = I^+_{\eta,\alpha+\beta}f$$
$$= I^+_{\eta,\beta}(I^+_{\eta+\beta,\alpha}f) = I^+_{\eta+\beta,\alpha}(I^+_{\eta,\beta}f).$$

This corresponds to K Theorem 4. The proof is similar, if more laborious.

There is a corresponding theorem for $K^-_{\zeta,\alpha}$.

THEOREM 4. *Let $\Re(\alpha) > m$ ($m = 1, 2, \ldots, h$), and let $f(x)$ belong to L_p. Then*

$$I^+_{\eta,\alpha}f = (-)^m I^+_{\eta+m,\alpha-m}(I^-_{\eta,m}f).$$

This is an easy corollary from Theorem 3. We have from Theorem 3

$$I^+_{\eta,\alpha}f = I^+_{\eta+m,\alpha-m}(I^+_{\eta,m}f),$$

and from this Theorem 4 follows by (8).

The particular case $m = h$ gives the decomposition (5) of $I^+_{\eta,\alpha}$ into h integrations from x to infinity and $a-h$ subsequent ones from zero to x.

There is a similar theorem on $K^-_{\zeta,\alpha}$.

7. The Mellin transform of the operations $I^+_{\eta,\alpha}$ and $K^-_{\zeta,\alpha}$

The Mellin transform $\mathfrak{M}_t f(x)$ of a function $f(x)$ belonging to L_p ($1 \leqslant p \leqslant 2$) is defined by

$$\mathfrak{M}_t f(x) = \int_0^\infty f(x)x^{it}\,dx \quad (p = 1) \tag{10}$$

and

$$\mathfrak{M}_t f(x) = \operatorname*{l.i.m.}_{X\to\infty}{}^{\text{index } p'} \int_{1/X}^{X} f(x)x^{it-1/p'}\,dx \quad (1 < p \leqslant 2). \tag{11}$$

The inverse Mellin transform $\mathfrak{M}^{-1}\phi(t)$ of a function belonging to $L_{p'}(-\infty,\infty)$ $(1 \leqslant p' \leqslant 2)$ is defined by

$$\mathfrak{M}^{-1}\phi(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi(t)x^{-it}\,dt \quad (p'=1) \tag{12}$$

and

$$\mathfrak{M}^{-1}\phi(t) = \frac{1}{2\pi}\operatorname*{index}_{T\to\infty} p\text{-l.i.m.} \int_{-T}^{+T} \phi(t)x^{-it-1/p}\,dt \quad (1<p'\leqslant 2). \tag{13}$$

Let M_p $(2 < p \leqslant \infty)$ be the set of all functions $f(x)$ belonging to L_p which are representable in the form $\mathfrak{M}^{-1}\phi(t)$, where $\phi(t)$ belongs to $L'_p(-\infty,\infty)$. In this case we call $\phi(t)$ the Mellin transform of $f(x)$ and write $\phi(t) = \mathfrak{M}_t f$. For functions belonging to M_p, (11) is valid even for $p > 2$.

THEOREM 5 (a). *Let $1 \leqslant p \leqslant 2$ and $f(x)$ belong to L_p. Then*

$$\mathfrak{M}_t I^+_{\eta,\alpha} f = \frac{\Gamma(\eta+1/p'-it)}{\Gamma(\eta+\alpha+1/p'-it)}\,\mathfrak{M}_t f \tag{14}$$

and

$$\mathfrak{M}_t K_{\zeta,\alpha} f = \frac{\Gamma(\zeta+1/p+it)}{\Gamma(\zeta+\alpha+1/p+it)}\,\mathfrak{M}_t f. \tag{15}$$

THEOREM 5 (b). *Let $2 < p \leqslant \infty$ and let $f(x)$ belong to M_p. Then $I^+_{\eta,\alpha} f$ and $K_{\zeta,\alpha} f$ also belong to M_p, and* (14) *and* (15) *hold.*

The proof of these theorems follows the lines of the proofs of Theorems 5 (a) and 5 (b) of K.

8. Fractional derivatives

THEOREM 6. *For the equation $g(x) = I^+_{\eta,\alpha} f(x)$ to have a solution $f(x)$ belonging to L_p, we have the conditions that $\Re(\eta+\alpha)+1/p' \neq 0, -1, -2, \ldots$, and that $g(x)$ belongs to L_p and $|t|^{\alpha}\mathfrak{M}_t g$ to $L_{p'}(-\infty,\infty)$. These conditions are*

(a) *necessary when $1 \leqslant p < 2$;*

(b) *necessary and sufficient when $p = 2$;*

(c) *sufficient when $2 < p \leqslant \infty$.*

The proof is essentially the same as that of K Theorem 7. The corresponding theorem holds for $K_{\zeta,\alpha}$.

II. Applications to Hankel Transforms

9. The operations $I^+_{\eta,\alpha}$ and $K^-_{\eta,\alpha}$ in L_2

Throughout the second part $p = p' = 2$ and h is a non-negative integer; in particular $h = 0$ when $\Re(\eta) > -\frac{1}{2}$, and $-\frac{1}{2} < \Re(\eta)+h < \frac{1}{2}$ when $\Re(\eta) < -\frac{1}{2}$. Negative integer values of $\Re(\eta)-\frac{1}{2}$ and $\Re(\eta+\alpha)-\frac{1}{2}$ are excluded. Empty sums are to be interpreted as zero.

We shall denote by $L_2^{(\alpha)}$ the set of functions $f(x)$ belonging to L_2 for which $|t|^{\max[0,-\Re(\alpha)]}\mathfrak{M}_t f$ belongs to $L_2(-\infty,\infty)$. Obviously $L_2^{(\alpha)}$ is identical with L_2 when $\Re(\alpha) \geqslant 0$, and $L_2^{(\alpha)}$ is a sub-set of L_2 when $\Re(\alpha) < 0$.

We can express our theorems more concisely in L_2 when defining the operations in the whole complex plane of the parameter α.

(i) If $\Re(\alpha) > 0$, we retain the definitions (1) and (2).

(ii) If $\Re(\alpha) = 0$, $\alpha = i\rho$ (ρ real), we define our operators by

$$I^+_{\eta,i\rho} f = \underset{\epsilon\to 0}{\text{l.i.m. sq.}}\, I^+_{\eta,i\rho+\epsilon} f, \qquad K^-_{\eta,i\rho} f = \underset{\epsilon\to 0}{\text{l.i.m. sq.}}\, K^-_{\eta,i\rho+\epsilon} f \quad (\epsilon > 0), \tag{16}$$

the limit being taken in the mean-square sense. In a future paper of Kober's it will be shown that when f belongs to L_2, then $I^+_{\eta,i\rho} f$ and $K^-_{\eta,i\rho} f$ exist almost everywhere in $(0,\infty)$ and belong to L_2.

(iii) If $\Re(\alpha) < 0$, we define $g = I^+_{\eta,\alpha} f$ and $h = K^-_{\eta,\alpha} f$ as the solutions (if any) of the integral equations

$$f = I^+_{\eta+\alpha,-\alpha} g, \qquad f = K^-_{\eta+\alpha,-\alpha} h. \tag{17}$$

Obviously, from this definition of the operators of fractional integration in the whole plane of the complex parameter α,

$$(I^+_{\eta,\alpha})^{-1} = I^+_{\eta+\alpha,-\alpha} \quad \text{and} \quad (K^-_{\eta,\alpha})^{-1} = K^-_{\eta+\alpha,-\alpha}$$

follow. Hence operation and inversion are not essentially different. For $\Re(\alpha) > 0$ we have fractional integrals; for $\Re(\alpha) < 0$ fractional derivatives.

The fundamental theorem on these extended operators is

Theorem 7. *If and only if $f(x)$ belongs to $L_2^{(\alpha)}$, then $I^+_{\eta,\alpha} f$ and $K^-_{\eta,\alpha} f$ exist almost everywhere in $(0,\infty)$ and belong to $L_2^{(-\alpha)}$.*

Proof. (i) $\Re(\alpha) > 0$. Here $L_2^{(\alpha)}$ is identical with L_2. Hence, in consequence of Theorem 1, $I^+_{\eta,\alpha} f$ and $K^-_{\eta,\alpha} f$ exist, and in consequence of Theorem 5 (a) they belong to $L_2^{(-\alpha)}$.

(ii) $\Re(\alpha) = 0$. Here the above-mentioned result of Kober's gives the statement of Theorem 7, for $L_2^{(i\rho)}$ as well as $L_2^{(-i\rho)}$ is identical with L_2.

(iii) $\Re(\alpha) < 0$. Here $L_2^{(-\alpha)}$ is identical with L_2 and hence the statement of Theorem 7 follows from Theorem 6.

Hence $L_2^{(\alpha)}$ can be characterized alternatively as the set of functions in L_2 which possess an integral of order α (or a derivative of order $-\alpha$).

10. The Hankel transform in L_2

Throughout the rest of the paper ν is supposed to be a complex parameter such that neither $\Re(\nu)$ nor $\Re(\nu+2\alpha)$ is a negative odd integer; n is a non-negative integer and $n = 0$ when $\Re(\nu) > -1$, and $-1 < \Re(\nu)+2n < 1$ when $\Re(\nu) < -1$. Also we put

$$J_{\nu,n}(z) = \sum_{s=n}^{\infty} \frac{(-)^s(\frac{1}{2}z)^{\nu+2s}}{s!\,\Gamma(\nu+s+1)},$$

and
$$\mathfrak{H}_\nu f(x) = \underset{Y\to\infty}{\text{l.i.m. sq.}} \int_0^Y J_{\nu,n}\{2\sqrt{(xy)}\}f(y)\,dy, \tag{18}$$

and call $F(x) = \mathfrak{H}_\nu f(x)$ the Hankel transform of $f(x)$.

It is well known that $\mathfrak{H}_\nu f$ exists almost everywhere in $(0,\infty)$ and belongs to L_2 if $f(x)$ belongs to L_2. Also

$$\mathfrak{M}_t F = \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}+it)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}-it)}\mathfrak{M}_{-t}f. \tag{19}$$

It is seen from this last relation that, $\Gamma(\frac{1}{2}\nu+\frac{1}{2}+it)/\Gamma(\frac{1}{2}\nu+\frac{1}{2}-it)$ being bounded, F belongs to $L_2^{(\alpha)}$ if f belongs to $L_2^{(\alpha)}$ and conversely.

11. Decomposition of $I^+_{\eta,\alpha}$ and $K^-_{\eta,\alpha}$ into Hankel transforms

We introduce the operator* $S_{\eta,\alpha}$, defined by

(i) If $\Re(\alpha) \geqslant 0$,

$$S_{\eta,\alpha}f = \underset{Y\to\infty}{\text{l.i.m. sq.}}\; x^{-\frac{1}{2}\alpha} \int_0^Y y^{-\frac{1}{2}\alpha} J_{2\eta+\alpha,h}\{2\sqrt{(xy)}\}f(y)\,dy. \tag{20}$$

When $f(x)$ belongs to L_2 and $\Re(\alpha) > \frac{1}{2}$, then the integral is convergent in the ordinary sense.

(ii) If $\Re(\alpha) < 0$, $\qquad S_{\eta,\alpha}f = (S_{\eta+\alpha,-\alpha})^{-1}f.$ (21)

* Cf. K § 7 and footnote. Our notation is slightly different from that of K.

If $f(x)$ belongs to $L_2^{(\alpha)}$, then $S_{\eta,\alpha}f$ exists almost everywhere in $(0,\infty)$, belongs to $L_2^{(-\alpha)}$, and

$$\mathfrak{M}_t S_{\eta,\alpha}f = \frac{\Gamma(\eta+\frac{1}{2}+it)}{\Gamma(\eta+\alpha+\frac{1}{2}-it)}\mathfrak{M}_{-t}f. \tag{22}$$

Obviously we have $S_{\frac{1}{2}\nu,0} = \mathfrak{H}_\nu$.

Both of our operations of fractional integration (or differentiation, as the case may be) can be represented by a product of two operations S.

THEOREM 8. *In addition to the conventions at the beginning of* § 9, *let* $\Re(\eta+\beta)-\frac{1}{2} \neq -1, -2, -3,\ldots$, *and suppose that*

$$\Re(\beta) \geqslant \min[0, \Re(\alpha)].$$

If $f(x)$ *belongs to* $L^{(\alpha)}$, *then*

$$I^+_{\eta,\alpha}f = S_{\eta+\beta,\alpha-\beta}(S_{\eta,\beta}f), \qquad K^-_{\eta,\alpha}f = S_{\eta,\alpha-\beta}(S_{\eta+\alpha-\beta,\beta}f). \tag{23}$$

The proof of this is very similar to that of K Theorem 8, (23) being the generalization of K (7.8). Under the hypotheses of Theorem 8, $f(x)$ also belongs to $L_2^{(\beta)}$ and hence $S_{\eta,\beta}f$ exists almost everywhere in $(0,\infty)$. Furthermore, it is seen from (22) that $S_{\eta,\beta}f$ belongs to $L_2^{(\alpha-\beta)}$. Hence $S_{\eta+\beta,\alpha-\beta}(S_{\eta,\beta}f)$ exists almost everywhere in $(0,\infty)$, and

$$\begin{aligned}\mathfrak{M}\{S_{\eta+\beta,\alpha-\beta}(S_{\eta,\beta}f)\} &= \frac{\Gamma(\eta+\beta+\frac{1}{2}+it)}{\Gamma(\eta+\alpha+\frac{1}{2}-it)}\mathfrak{M}_{-t}\{S_{\eta,\beta}f\} \\ &= \frac{\Gamma(\eta+\beta+\frac{1}{2}+it)}{\Gamma(\eta+\alpha+\frac{1}{2}-it)}\,\frac{\Gamma(\eta+\frac{1}{2}-it)}{\Gamma(\eta+\beta+\frac{1}{2}+it)}\mathfrak{M}_t f = \mathfrak{M}_t\{I^+_{\eta,\alpha}f\}.\end{aligned}$$

The first equation of (23) follows; the proof of the second is similar.

12. Connexion between Hankel transforms of different order

THEOREM 9. *Let* $f(x)$ *belong to* $L_2^{(\alpha)}$ *and*

(a) *let* $F = \mathfrak{H}_\nu f$, $g = I^+_{\frac{1}{2}\nu,\alpha}f$, $G = I^+_{\frac{1}{2}\nu,\alpha}F$: *then* $G = \mathfrak{H}_{\nu+2\alpha}g$;

(b) *let* $F = \mathfrak{H}_{\nu+2\alpha}f$, $g = K^-_{\frac{1}{2}\nu,\alpha}f$, $G = K^-_{\frac{1}{2}\nu,\alpha}F$: *then* $G = \mathfrak{H}_\nu g$.

This theorem covers all the theorems of E-K which are concerned with the connexion of Hankel transforms of different order in L_2.

To prove Theorem 9 we remark that, $L_2^{(\alpha)}$ being a sub-set of L_2, $\mathfrak{H}_\nu f$ and $\mathfrak{H}_{\nu+2\alpha}f$ respectively exist almost everywhere in $(0,\infty)$ and belong to L_2. From (19) it is seen that $\mathfrak{H}_\nu f$ and $\mathfrak{H}_{\nu+2\alpha}f$ even belong to $L_2^{(\alpha)}$. Theorem 7 ensures the existence of g and G almost everywhere in $(0,\infty)$. Further, g, G—in the respective cases (a), (b)—

belong to $L_2^{(-\alpha)}$ and hence to L_2. The rest of the proof follows the lines of that of E-K Theorem 5.

A few particular cases of Theorem 9 are of some interest.

(i) When $\Re(\nu) > -1$ and $\Re(\alpha) > 0$, we have Kober's operators, and this particular case is identical with E-K Theorem 5.

(ii) When $\Re(\nu+2\alpha) > -1$ and $\Re(\alpha) < 0$, then, by (17), our operators are inversions of Kober's operators. In this case Theorem 9, except for the notation, is identical with E-K Theorem 6.

(iii) $\Re(\nu) < -1$ and $\alpha = m = 1, 2, \ldots, n$. In this case

$$I^{+}_{\frac{1}{2}\nu,\alpha} = (-)^{\alpha} I^{-}_{\frac{1}{2}\nu,\alpha} \quad \text{and} \quad K^{-}_{\frac{1}{2}\nu,\alpha} = (-)^{\alpha} K^{+}_{\frac{1}{2}\nu,\alpha},$$

and we have E-K Theorem 3, as far as it is concerned with L_2.

(iv) $\Re(\nu) < -1$ and $\alpha = -m = -1, -2, \ldots, -n$. Here, except for the notation, we have E-K Theorem 7.

The possibility of condensing simply all these theorems in a more general one (which is more than the mere sum of its special cases i–iv) speaks, in my opinion, in favour of the extension of the operators.

CORRECTION

On Diophantine approximation and the generalized elliptic theta function

H. S. A. POTTER (*Aberdeen*)

[Received 19 April 1940]

In my paper under this title* Theorem 10† is false. The correct result is

$$\sum_{n=0}^{\infty} \{A(\mathfrak{S}, n)\}^2 e^{-yn} = \lambda y^{1-m} + O(y^{2-m}) + O(y^{-\frac{1}{2}m})$$

for $m > 4$, where λ depends on $\mathfrak{S}$ only, as has been proved by A. Walfisz.‡

* *Quart. J. of Math.* (Oxford), 9 (1938), 161–75.

† Loc. cit. 174.

‡ 'Über Gitterpunkte in mehrdimensionalen Ellipsoiden', *Cas. mat. fys.* 66 (1936), 1–19.

BRITISH ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE

MATHEMATICAL TABLES

VOLUME IX

TABLES OF POWERS

Giving Integral Powers of Integers

Initiated by J. W. L. GLAISHER

Demy 4to 15s. net

Published for the British Association

CAMBRIDGE UNIVERSITY PRESS

The Indian Mathematician

RAMANUJAN

Twelve lectures on subjects suggested by his life and work

By G. H. HARDY With portrait 25s. net

This is the story of Ramanujan, who, as Professor Hardy wrote in the Preface to his collected papers, 'would probably have been a greater mathematician if he had been caught and tamed a little in his youth; he would have discovered more that was new, and that, no doubt, of greater importance. On the other hand he would have been less of a Ramanujan and more of a European professor, and the loss might have been greater than the gain.'

CAMBRIDGE UNIVERSITY PRESS

Frankly recognizing that, to those whom nature did not initiate, there is a mystery about mathematics,

Professor G. H. HARDY

discusses its beauty and power and gives a personal account of what mathematics has meant to him as a man.

A MATHEMATICIAN'S APOLOGY

3s. 6d. net

Cambridge University Press

Ready in January 1941

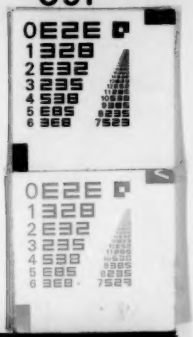
ALGEBRA

A Text-Book of Determinants, Matrices, and Algebraic Forms

By W. L. FERRAR

12s. 6d. *net*

Written primarily for undergraduates, it includes such parts of the theory of determinants, matrices, and algebraic forms as are appropriate to a general honours course in mathematics; it excludes topics that are appropriate only to highly specialized or to post-graduate courses of study.

OXFORD UNIVERSITY PRESS

C51